Fundamental groups of small covers

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The 5th Korea Toric Topology Winter Workshop

Gyeongju, Korea, January 21-23, 2019
1. Introduction

2. Presentations of Fundamental Groups

3. Main Results and Applications
An $n$-dimensional **small cover** is a closed $n$-manifold $M$ with a locally standard $\mathbb{Z}_2^n$-action whose orbit space is a simple polytope $P$.

$$\pi : M \longrightarrow P$$
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The $\mathbb{Z}_2^n$-action on $M$ determines a $\mathbb{Z}_2^n$-valued characteristic function $\lambda$ on the set of facets of $P$

$$\lambda : \mathcal{F}(P) \triangleq \{F_1, F_2, \cdots, F_m\} \longrightarrow \mathbb{Z}_2^n$$
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$$\lambda : \mathcal{F}(P) \triangleq \{F_1, F_2, \cdots, F_m\} \longrightarrow \mathbb{Z}_2^n$$

such that

$$\forall f = F_1 \cap F_2 \cap \cdots \cap F_k,$$

$$G_f \triangleq \langle \lambda(F_1), \lambda(F_2), \cdots, \lambda(F_k) \rangle \cong \mathbb{Z}_2^k.$$
Small cover

\[ M = P \times \mathbb{Z}_2^n / \sim \]

where \((p, g) \sim (q, h)\) iff \(p = q, \ g^{-1}h \in G_{f(p)}\), and \(f(p)\) is the unique face of \(P\) that contains \(p\) in its relative interior, \(G_{f(p)} = \{1\}\) if \(p \in P^\circ\).
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where $(p, g) \sim (q, h)$ iff $p = q$, $g^{-1}h \in G_{f(p)}$, and $f(p)$ is the unique face of $P$ that contains $p$ in its relative interior, $G_{f(p)} = \{1\}$ if $p \in P^\circ$.

Define right-angled Coxeter group of $P$

$W = \langle s_F, \forall F \in \mathcal{F}(P) \mid s_F^2 = 1; (s_F s_{F'})^2 = 1, F \cap F' \neq \emptyset \rangle$
The Borel construction of $\mathbb{Z}_2^n$ on $M$

$$BP = M \times_{\mathbb{Z}_2^n} E\mathbb{Z}_2^n$$

where $E\mathbb{Z}_2^n = (S^\infty)^n$. And $\pi_1(BP) \cong W$. 
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$M \rightarrow BP \rightarrow B\mathbb{Z}_2^n$ induces a right-split exact sequence

$$
1 \rightarrow \pi_1(M) \rightarrow W \xleftarrow{\phi} \mathbb{Z}_2^n \xrightarrow{\gamma} 1
$$

where $\phi(s_F) = \lambda(F), \forall F \in \mathcal{F}(P)$. 

Cell decomposition
Generators and relations

Cell-1

Relation-1: \( x_{F,g} x_{F,g\lambda(F)} = 1 \)

Relation-2: \( x_{F,g} x_{F',g\lambda(F)} = x_{F',g} x_{F,g\lambda(F')} \)
Generators and relations

Relation-2: \( x_{F,g} x_{F',g} \lambda(F) = x_{F',g} x_{F,g} \lambda(F') \)

Relation-3: \( x_{F,g} = 1, \ p_0 \subset F \)
Presentation of $\pi_1(M, p_0)$

\[
\pi_1(M, p_0) = \langle x_{F,g}, \forall F, g \mid x_{F,g} x_{\lambda(F)} = 1; \\
x_{F,g} x_{F',g} x_{\lambda(F)} = x_{F',g} x_{F,g} x_{\lambda(F')}, \quad F \cap F' \neq \emptyset; \\
x_{F,g} = 1, \quad p_0 \in F \rangle
\]
Relation between $\pi_1(M)$ and $W$

$\tilde{M} = Q \times \pi_1(M)/ \sim = P \times W/ \sim$

$\textbf{Rk: } \lambda(F_1, F_2, F, F') = (e_1, e_2, e_1e_2, e_2).$
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$\chi_{F,1}(Q, 1) \mapsto (Q, x_{F,1})$
Relation between $\pi_1(M)$ and $W$

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$F$

$F_1$

$F$

$F_1$

$F$

$F_2$

$x_{F,1}(Q, 1) \mapsto (Q, x_{F,1})$

$s_F(P, 1) \mapsto (P, s_F)$

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$s_F(P, 1) \mapsto (P, s_F)$

$x_{F,1}(P, 1) \mapsto (P, s_{F_2}s_{F_1}s_F)$

$\gamma(\lambda(F)) \cdot s_F(P, 1) \mapsto s_{F_2}s_{F_1}s_F(P, 1)$

Rk: $\lambda(F_1, F_2, F, F') = (e_1, e_2, e_1e_2, e_2)$. 
Relation between $\pi_1(M)$ and $W$

$$\alpha : \pi_1(M, p_0) \rightarrow W$$

$$x_{F, g} \rightarrow \gamma(g\lambda(F)) \cdot \gamma(\lambda(F))s_F \cdot (\gamma(g\lambda(F)))^{-1}$$

$$= \gamma(g)s_F\gamma(g\lambda(F))$$
Relation between $\pi_1(M)$ and $W$

\[ \alpha : \pi_1(M, p_0) \longrightarrow W \]

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\[ = \gamma(g)s_F\gamma(g\lambda(F)) \]

\[ \begin{array}{c}
1 \longrightarrow \pi_1(M) \xrightarrow{\alpha} W \xrightarrow{\phi} \mathbb{Z}_2^n \longrightarrow 1
\end{array} \quad (1) \]
Idea

\[ M \quad \pi_1(M) \]

\[ P \quad W \]

\[ \pi \quad \alpha \]
Idea

\[
\begin{array}{ccc}
M & \xrightarrow{\pi} & \pi_1(M) \\
\downarrow{\pi} & & \downarrow{\alpha}
\end{array}
\]

\[
\begin{array}{ccc}
P & \xrightarrow{\pi_1(M)} & W
\end{array}
\]
Idea
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Idea

- $M$ to $\pi_1(M)$
- $\pi$ to $\alpha$
- $P$ to $W$
- Hard to easy

For any proper face $f$ of $P$,

- Define $\mathcal{F}(f^\perp) \triangleq \{ F \in \mathcal{F}(P) \mid \text{dim}(f \cap F) = \text{dim}(f) - 1 \}$. So $\mathcal{F}(f^\perp)$ consists of those facets of $P$ that intersect $f$ transversely.
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- A submanifold $\Sigma$ in $M$ is called $\pi_1$-injective if the inclusion $\Sigma \hookrightarrow M$ induces a monomorphism in the fundamental group.
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**Theorem (Wu-Yu, 2017)**

Let $M$ be a small cover over a simple polytope $P$ and $f$ be a proper face of $P$. The following two statements are equivalent.

- The facial submanifold $M_f$ is $\pi_1$-injective.
- For any $F, F' \in \mathcal{F}(f^\perp)$, we have $f \cap F \cap F' \neq \emptyset$ whenever $F \cap F' \neq \emptyset$. 
A simple polytope $P$ is called flag if a collection of facets of $P$ has common intersection whenever they pairwise intersect.
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**Proposition (Wu-Yu, 2017)**

Let $M$ be a small cover over $P$. Then $P$ is flag if and only if every facial submanifold of $M$ is $\pi_1$-injective.
For a 3-dimensional simple polytope $P$,

- A $k$-circuit in $P$ is a simple loop on the boundary of $P$ which intersects transversely with the interior of exactly $k$ distinct edges,
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- A **$k$-belt** in $P$ is a set of $k$ distinct faces $F_1, \cdots, F_k$ of $P$ such that $F_i \cap F_{i+1} \neq \emptyset$ for $1 \leq i \leq k-1$, $F_k \cap F_1 \neq \emptyset$, and any three faces in the belt have no common intersection.
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- Each $k$-belt can determine a prismatic $k$-circuit.
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- Each $k$-belt can determine a prismatic $k$-circuit. A prismatic 3-circuit determines a 3-belt; and if there is no prismatic 3-circuit, then a prismatic 4-circuit determines a 4-belt.
Let $M$ be a 3-small cover over $P(\neq \Delta^3)$, the following facts was referred in DJ’s paper(corrected):

- If there exist prismatic 3-circuits in $P$, then $M$ can be decomposed into prime pieces glued along $S^2$ or $\mathbb{R}P^2$. 
Applications

Let $M$ be a 3-small cover over $P(\neq \Delta^3)$, the following facts was referred in DJ’s paper (corrected):

- If there exist prismatic 3-circuits in $P$, then $M$ can be decomposed into prime pieces glued along $S^2$ or $\mathbb{RP}^2$.

- If there is no prismatic 3-circuit but prismatic 4-circuits in $P$, then $M$ can be decomposed into atoroidal and Seifert fibered pieces glued along tori or Klein bottles.
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- If there is no prismatic 3-circuit but prismatic 4-circuits in $P$, then $M$ can be decomposed into atoroidal and Seifert fibered pieces glued along tori or Klein bottles.
- If there is no prismatic 3 or 4-circuit in $P$, then $M$ is hyperbolic.
Let $M$ be a connected 3-manifold.

- $M$ is called **prime** if $M = M_1 \# M_2$ implies $M_1 = S^3$ or $M_2 = S^3$. 
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- $M$ is called **prime** if $M = M_1 \# M_2$ implies $M_1 = S^3$ or $M_2 = S^3$.

- $M$ is called **irreducible** if every embedded 2-sphere bounds a 3-ball. An prime 3-manifold is irreducible except $S^2$-bundle over $S^1$. 

**Theorem (Kneser, Milnor, Prime Decomposition Theorem)**

Each compact 3-manifold $M$ can factor as a connected sum of prime manifolds. This decomposition is unique under the assumption that $M$ is orientable.
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Proposition

Let $M$ be a 3-dimensional small cover over a simple polytope $P$, then $M$ is irreducible if and only if there is no prismatic 3-circuit in $P$. 
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In particular, the prime decomposition of an oriented 3-small cover is equivalent to cutting surgery along prismatic 3-circuits in $P$. 
A compact 3-manifold $M$ is called atoroidal if it contains no essential torus.
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A manifold $M$ is called \textit{hyperbolic} if it admits a complete Riemannian metric of constant sectional curvature $-1$. 
A compact 3-manifold $M$ is called atoroidal if it contains no essential torus.

A manifold $M$ is called hyperbolic if it admits a complete Riemannian metric of constant sectional curvature $-1$.

**Theorem (Jaco-Shalen-Johannson, Torus Decomposition Theorem)**

For an oriented, irreducible, closed 3-manifold, there exists a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \cdots, T_m$ such that each component of $M$ cut along $T_1 \cup \cdots \cup T_m$ is atoroidal or Seifert fibered, and a minimal such collection $T_1, \cdots, T_m$ is unique up to isotopy.
Theorem (Perelman, Geometrization Theorem)

Let $M$ be a irreducible closed 3-manifold. There exists a (possibly empty) collection of disjointly embedded incompressible surface $S_1, \ldots, S_m$ which are either tori or Klein bottles, such that each component of $M$ cut along $S_1 \cup \cdots \cup S_m$ is geometric. Any such collection of tori with a minimal number of components is unique up to isotopy.
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Proposition

Let $M$ be a 3-small cover over a simple polytope $P$, then $M$ is atoroidal if and only if there is no 4-belt in $P$. In particular, the JSJ decomposition or geometric decomposition of a irreducible 3-small cover is equivalent to cutting surgery along prismatic 4-circuits in $P$. 
Theorem (Thurston, Hyperbolization Theorem)

Each irreducible, atoroidal, closed 3-manifold with infinite fundamental group is hyperbolic.
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Each irreducible, atoroidal, closed 3-manifold with infinite fundamental group is hyperbolic.

**Proposition**

Let $M$ be a 3-small cover over a simple polytope $P(\neq \Delta^3)$, then $M$ is hyperbolic if and only if there is no prismatic 3 or 4-circuit in $P$. 
Proposition (Wu-Yu, 2017)

A small cover $M$ over a simple 3-polytope $P$ can admit a Riemannian metric with nonnegative scalar curvature if and only if $P$ is combinatorially equivalent to the cube $[0, 1]^3$ or a polytope obtained from $\Delta^3$ by a sequence of vertex cuts.
A classification for 3-small cover

3-belt (pris 3-circuit)

- YES
  - reducible
    - oriented
      - Prime decomposition
        - $\text{vc}^k(\Delta^3), k \geq 1$
        - > 0 scalar curvature
      - ONLY
  - irreducible
    - spherical
      - $\mathbb{R}P^3(\Delta^3)$
  - atoroidal
    - Real Bott($I^3$)
      - flat
      - NO
        - $|\pi_1| = \infty$
  - hyperbolic
    - $|\pi_1| < \infty$

4-belt

- YES
  - toroidal
    - JSJ or geometric decomposition
    - ONLY
  - no 3-belt

ONLY
End of Talk

The 5th Korea Toric Topology Winter Workshop

Gyeongju, Korea. 10:30 - 11:10 January 22, 2019
Some references

- Buchstaber and Panov, Torus actions and their applications in topology and combinatorics. (2002).
- Kapovich, Hyperbolic manifolds and discrete groups. (2010).
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