Towards transverse toric geometry

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The main concerns:

1. complex manifold $M$ with a maximal action of compact torus,
2. canonical foliation $F$ on $M$.

Purpose
The foliated manifold $(M, F)$ behaves similar to a toric variety.
$M$ : connected manifold,
$G$ : compact torus, $G \curvearrowright M$ : effective.

For $x \in M$, $T_x M$ is a faithful representation of $G_x$.

$$T_x M \cong T_x (G \cdot x) \oplus T_x M / T_x (G \cdot x).$$

In particular, $G_x \to GL(T_x M / T_x (G \cdot x))$ is injective.

$$\dim G_x = \dim G_x^0 \leq \frac{1}{2} (\dim M - \dim G \cdot x).$$

$$\Longrightarrow \dim G + \dim G_x \leq \dim M.$$
Maximal actions

Definition

$M$ : connected manifold, 
$G$ : compact torus, $G \acts M$ : effective.
$G \acts M$ is maximal if

$$\exists x \in M \text{ s.t. } \dim G + \dim G_x = \dim M.$$  

Remark

$G \acts M$ maximal $\implies G$ is a maximal compact torus in Diff $M$. 


\( M: \) compact connected complex manifold, 
\( G \subset \text{Aut}(M, J): \) maximal compact torus.

\( g \subset \mathfrak{X}(M, J). \) Put

\[ h' := g \cap Jg \subset g, \quad H' := \exp h'. \]

**Proposition**

The followings hold:

1. \( H' \) is a complex Lie group.
2. \( H' \curvearrowright M: \) holomorphic and local free.
3. \( H' \) does not depend on the choice of \( G. \)

**Definition**

The canonical foliation \( F \) on \( M \) is a foliation whose leaves are \( H' \)-orbits.
Example (complex tori)

- $M = \mathbb{C}^n / \Gamma$, $\Gamma : \mathbb{C}^n$ lattice.
- $G = M = \mathbb{C}^n / \Gamma \cong (S^1)^{2n}$.
- For all $x \in M$,
  \[
  \dim G + \dim G_x = \dim M.
  \]
  
  \[
  \begin{array}{c}
  \underbrace{\dim G}_{2n} + \underbrace{\dim G_x}_{0} = \underbrace{\dim M}_{2n}.
  \end{array}
  \]

- $\mathfrak{h'} = \mathfrak{g} \cap J\mathfrak{g} = \mathfrak{g}$ because any fundamental vector field is $J$ of a fundamental vector field.

- The leaf space $M/F$ is a point, i.e. 0-dimensional toric variety.
Example (toric varieties)

- $M$: nonsingular complete toric variety of dimension $n$,
- $G = (S^1)^n \subset (\mathbb{C}^\times)^n \curvearrowright M$.
- For $x \in M^G$,
  $$\underbrace{\dim G}_n + \underbrace{\dim G_x}_n = \underbrace{\dim M}_{2n}.$$
- $\mathfrak{h}' = \mathfrak{g} \cap J\mathfrak{g} = \{0\}$, i.e. every leaf of $F$ is a point.
- The leaf space $M/F$ is nothing but a toric variety $M$. 
Example (Hopf surfaces)

- \( \alpha_1, \alpha_2 \in \mathbb{C} \), \( 1 < |\alpha_1| \leq |\alpha_2| \).
- \( \Gamma = \{(\alpha_1^k, \alpha_2^k) \in (\mathbb{C}^\times)^2 \mid k \in \mathbb{Z}\} \subset (\mathbb{C}^\times)^2 \). \( \Gamma \cong \mathbb{Z} \).
- \((\mathbb{C}^\times)^2 \acts \mathbb{C}^2 \setminus \{0\} \) via \((g_1, g_2) \cdot (z_1, z_2) = (g_1z_1, g_2z_2)\).
- \( M_{\alpha_1, \alpha_2} := (\mathbb{C}^2 \setminus \{0\})/\Gamma \) Hopf surface.
- \( G : \) maximal compact torus in \((\mathbb{C}^\times)^2/\Gamma \acts M \). \( G \cong (S^1)^3 \).
- For \( x = [1, 0] \in M_{\alpha_1, \alpha_2} \),

\[
\dim G + \dim G_x = \dim M_{\alpha_1, \alpha_2}.
\]

- \( \mathfrak{h}' \subset \mathfrak{g} \) is 2-dimensional subspace.
- \( H' \subset G \) is a subtorus iff \( \alpha_1^{n_1} = \alpha_2^{n_2} \) for some \( n_1, n_2 \in \mathbb{N} \).
- \( M/F \) is Hausdorff iff \( \alpha_1^{n_1} = \alpha_2^{n_2} \) for some \( n_1, n_2 \in \mathbb{N} \).
$\mathcal{C}_1$: category of compact connected complex manifolds with maximal actions.

- An object of $\mathcal{C}_1$ is $(M, G, y)$:
  - $G$: compact torus,
  - $M$: compact connected complex manifold,
  - $G \curvearrowright M$ maximal, preserving the complex structure.
  - $y \in M$ such that $G_y = \{1_G\}$.

- For $(M_1, G_1, y_1), (M_2, G_2, y_2)$, $(f, \alpha) \in \text{Hom}_{\mathcal{C}_1}((M_1, G_1, y_1), (M_2, G_2, y_2))$ if
  - $\alpha: G_1 \rightarrow G_2$ is a smooth homomorphism,
  - $f: M_1 \rightarrow M_2$ is an $\alpha$-equivariant holomorphic map,
  - $f(y_1) = y_2$. 
\( C_2 \) : category.

- An object of \( C_2 \) is \((\Delta, \mathfrak{h}, G)\):
  - \( G \) is a compact torus,
  - \( \Delta \) is a nonsingular fan in \( \mathfrak{g} \) with respect to the lattice \( \ker \exp_G \),
  - \( \mathfrak{h} \subset \mathfrak{g}^C = \mathfrak{g} \otimes \mathbb{C} \) is a \( \mathbb{C} \)-subspace

satisfying

- For the projection \( p : \mathfrak{g}^C \to \mathfrak{g} \),
  \[ p|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{g} \text{ is injective.} \]
- \( \mathfrak{h}' := p(\mathfrak{h}) \). The quotient map \( q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}' \) sends \( \Delta \) to a complete fan in \( \mathfrak{g}/\mathfrak{h}' \).

- For \((\Delta_1, \mathfrak{h}_1, G_1), (\Delta_2, \mathfrak{h}_2, G_2)\), \( \alpha \in \text{Hom}_{C_2}((\Delta_1, \mathfrak{h}_1, G_1), (\Delta_2, \mathfrak{h}_2, G_2)) \) if
  - \( \alpha : G_1 \to G_2 \) is a smooth homomorphism,
  - \( d\alpha : \mathfrak{g}_1 \to \mathfrak{g}_2 \) induces a morphism of fans \( d\alpha : \Delta_1 \to \Delta_2 \),
  - \( d\alpha^C : \mathfrak{g}_1^C \to \mathfrak{g}_2^C \) satisfies \( d\alpha^C(\mathfrak{h}_1) \subset \mathfrak{h}_2 \).
For \((M, G, y) \in \mathcal{C}_1\), we can
- get a nonsingular fan \(\Delta\) by an argument similar to toric geometry.
- get a \(\mathbb{C}\)-subspace \(\mathfrak{h} \subset \mathfrak{g}^\mathbb{C}\). \(H \subset G^\mathbb{C} \curvearrowright M\) : global stabilizers.
- verify \((\Delta, \mathfrak{h}, G) \in \mathcal{C}_2\).

For \((\Delta, \mathfrak{h}, G) \in \mathcal{C}_2\),
- \(X(\Delta)/H\) is a compact connected complex manifold.
- \(G \curvearrowright X(\Delta)\) descends to a maximal action \(G \curvearrowright X(\Delta)/H\).

**Theorem**

\(\mathcal{C}_1\) and \(\mathcal{C}_2\) are equivalent as categories.
**Definition**

Foliated manifolds \((M_1, F_1)\) and \((M_2, F_2)\) are **transverse equivalent** if there exist a foliated manifold \((M_0, F_0)\) and (holomorphic or smooth) maps \(f_i : M_0 \to M_i\) such that

1. \(f_i\) is a surjective submersion,
2. \(f_i^{-1}(x_i)\) is connected for all \(x_i \in M_i\),
3. the preimage of each leaf of \(F_0\) by \(f_i\) is a leaf of \(F_i\).

\[
\begin{array}{c}
M_0 \\
\downarrow f_1 \quad \downarrow f_2 \\
M_1 \quad \quad M_2
\end{array}
\]
Problem

\((M_1, G_1, y_1), (M_2, G_2, y_2) \in C_1\).

\(F_i: \) canonical foliation on \(M_i, \ i = 1, 2\).

When are \((M_1, F_1)\) and \((M_2, F_2)\) transversely equivalent?

Definition

\((M_1, G_1, y_1), (M_2, G_2, y_2) \in C_1\) are principal equivalent if there exists \((M_0, G_0, y_0) \in C_1\) and morphisms

\((f_i, \alpha_i) \in \text{Hom}_{C_1}((M_0, G_0, y_0), (M_i, G_i, y_i))\) such that

1. \(f_i: M_0 \to M_i\) is a principal \(\ker \alpha_i\)-bundle,
2. \(\ker \alpha_i\) is connected.

Proposition (sufficient condition)

If \((M_1, G_1, y_1)\) and \((M_2, G_2, y_2)\) are principal equivalent, then \((M_1, F_1)\) and \((M_2, F_2)\) are transversely equivalent.
Definition

A marked fan is a quadruple $(\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda})$, where

- $\tilde{V}$ is a finite dimensional $\mathbb{R}$-vector space;
- $\tilde{\Gamma}$ is a finitely generated subgroup of $\tilde{V}$ that spans $\tilde{V}$ $\mathbb{R}$-linearly;
- $\tilde{\Delta}$ is a fan in $\tilde{V}$ and each 1-cone is generated by an element of $\tilde{\Gamma}$;
- $\tilde{\lambda}$ is a function $\tilde{\lambda} : \tilde{\Delta}^{(1)} \rightarrow \tilde{\Gamma}$, where $\tilde{\lambda}(\rho)$ is a generator of $\rho \in \tilde{\Delta}^{(1)}$.

$(\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda}) :$ simplicial $\iff$ $\Delta :$ simplicial.

$(\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda}) :$ complete $\iff$ $\Delta :$ complete.
\( \tilde{C}_2 \) : class of all complete simplicial marked fans.

To \((M, G, y) \in C_1\), we can assign \((\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda}) \in \tilde{C}_2\). Let \((\Delta, h, G) \in C_2\) be the counterpart to \((M, G, y)\).

- \(\tilde{V} := g/p(h)\),
- \(\tilde{\Gamma} := q(\ker \exp_G) \subset \tilde{V}\),
- \(\tilde{\Delta} := q(\Delta)\),
- \(\tilde{\lambda} = q \circ \lambda\).

We get \(\tilde{F}_1 : C_1 \to \tilde{C}_2\).
Definition

\((\tilde{V}_1, \tilde{\Gamma}_1, \tilde{\Delta}_1, \tilde{\lambda}_1)\) and \((\tilde{V}_2, \tilde{\Gamma}_2, \tilde{\Delta}_2, \tilde{\lambda}_2)\) are isomorphic \iff \exists \varphi: \tilde{V}_1 \rightarrow \tilde{V}_2 \text{ linear isomorphism s.t.}

1. \(\varphi(\tilde{\Gamma}_1) = \tilde{\Gamma}_2,\)
2. \(\varphi(\tilde{\Delta}_1) = \tilde{\Delta}_2,\)
3. \(\varphi \circ \lambda_1 = \lambda_2 \circ \varphi.\)

Example (Hopf surface)

\(M_{\alpha_1, \alpha_2}\): Hopf surface.

The marked fan corresponding to \(M_{\alpha_1, \alpha_2}\) is isomorphic to \((\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda}), \text{ where } \tilde{V} = \mathbb{R},\)

\[\tilde{\Gamma} = \langle \log |\alpha_1|, \log |\alpha_2|, \log |\alpha_1| \frac{\arg \alpha_2}{2\pi} - \log |\alpha_2| \frac{\arg \alpha_1}{2\pi} \rangle, \ldots\]
Theorem

\((M_1, G_1, y_1), (M_2, G_2, y_2) \in C_1\) are principal equivalent iff 
\(\widetilde{\mathcal{F}}_1(M_1, G_1, y_1)\) and \(\widetilde{\mathcal{F}}_1(M_2, G_2, y_2)\) are isomorphic.

Theorem

\(\widetilde{\mathcal{F}}_1 : C_1 \to \widetilde{C}_2\) is essentially surjective.

Namely, \(\forall (\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda}) \in \widetilde{C}_2, \exists (M, G, y) \in C_1\) s.t. \(\widetilde{\mathcal{F}}_1(M, G, y)\) is isomorphic to \((\tilde{V}, \tilde{\Gamma}, \tilde{\Delta}, \tilde{\lambda})\).

\[
\begin{align*}
\text{principal equivalence} & \quad \iff \quad \text{marked fan isomorphism} \\
\downarrow & \\
\text{transverse equivalence}
\end{align*}
\]
\[(M, G, y) \in C_1, (\widetilde{V}, \widetilde{\Gamma}, \widetilde{\Delta}, \widetilde{\lambda}) = \tilde{F}_1(M, G, y).\]

**Theorem**

The leaf space \(M/F\) is a toric orbifold iff \(\text{rk} \widetilde{\Gamma} = \dim \widetilde{V}\).

**Theorem**

\(M\) is transverse Kähler w.r.t. \(F\) iff \(\widetilde{\Delta}\) is polytopal.

**Theorem**

\[H^*_B(M) \cong \mathbb{R}[x_1, \ldots, x_m]/\mathcal{I} + \mathcal{J},\]

\(\mathcal{I}\) : Stanley-Reisner ideal for the underlying simplicial cpx. of \(\widetilde{\Delta}\),
\(\mathcal{J}\) : linear ideal determined by \(\widetilde{\lambda}\).