Equivariant Cohomology of Torus Orbifolds

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Joint work with S. Kuroki and J. Song

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Toric and Quasitoric Manifolds

Toric Manifolds $\leftrightarrow$ comp. reg. fans $(\mathbb{C}^*)^n \odot X^{2n} \leftrightarrow \Sigma \subseteq \mathbb{R}^n$

comp. non-sing. toric varieties

$\loc. \ std. \ M/T^n \sim = P^n_{\lambda}: F(P) \rightarrow \mathbb{Z}^n$

P is a simple polytope satisfies basis condition
Toric and Quasitoric Manifolds

Toric Manifolds $\leftrightarrow$ comp. reg. fans $(\mathbb{C}^*)^n \bowtie X^{2n} \leftrightarrow \Sigma \subseteq \mathbb{R}^n$
comp. non-sing. toric varieties

Quasitoric Manifolds $\leftrightarrow$ Characteristic Pairs
$T^n \bowtie M^{2n} \leftrightarrow (P, \lambda)$
loc. std. st. $M/T^n \cong P^n$ $\lambda: \mathcal{F}(P) \rightarrow \mathbb{Z}^n$
P is a simple polytope satisfies basis condition
Equivariant Cohomology

Set \( T := T^n = S^1 \times \cdots \times S^1 \) and let

\[ ET \longrightarrow BT \]

be the universal \( T \)-bundle.

If \( T \circ M \), then we get a fibration

\[ M \longrightarrow ET \times_T M \longrightarrow BT. \]
Equivariant Cohomology

Set $T := T^n = S^1 \times \cdots \times S^1$ and let

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be the universal $T$-bundle.

If $T \circlearrowleft M$, then we get a fibration

$$M \to ET \times_T M \to BT.$$
Equivariant Cohomology

The equivariant cohomology of toric and quasitoric manifolds can be written in terms of the face rings of the corresponding combinatorial objects.

**Theorem (Danilov ’78 & Jurkiewicz ’85)**

\[ H_T^*(X(\Sigma)) \cong \mathbb{Z}[\Sigma] \]

**Theorem (Davis & Januszkiewicz ’91)**

\[ H_T^*(M(P,\lambda)) \cong \mathbb{Z}[P] \]
Equivariant Cohomology

The equivariant cohomology of toric and quasitoric manifolds can be written in terms of the face rings of the corresponding combinatorial objects.

Theorem (Danilov ’78 & Jurkiewicz ’85)

\[ H_T^*(X(\Sigma)) \cong \mathbb{Z}[\Sigma] \]

Theorem (Davis & Januszkiewicz ’91)

\[ H_T^*(M(P, \lambda)) \cong \mathbb{Z}[P] \]

Their non-equivariant cohomology rings,

\[ H^*(X(\Sigma)) \cong \mathbb{Z}[\Sigma]/\mathcal{J}_\Sigma \quad \& \quad H^*(M(P, \lambda)) \cong \mathbb{Z}[P]/\mathcal{J}_\lambda, \]

are given by factoring out by linear relations.
Properties

- Torus actions are locally standard
- Quotient $M/T$ is a *manifold with corners* (locally like $\mathbb{R}^n$)
- Isolated fixed points
- Can be rebuilt using combinatorial data: $T \times P/\sim$
- Cohomology is generated in degree 2 and hence concentrated in even degrees
- No torsion
\[
\mathbb{CP}^n = M(\Delta^n, (I_n \mid -1))
\]

**Example**

*Figure: The fan of \( \mathbb{CP}^2 \)*

\[
H_T^*(\mathbb{CP}^2(\Sigma)) \cong \mathbb{Z}[v_1, v_2, v_3]/(v_1 v_2 v_3) \cong H_T^*(M(\Delta^2, \lambda)) \\
H^*(\mathbb{CP}^2(\Sigma)) \cong \mathbb{Z}[v_1, v_2, v_3]/(v_1 v_2 v_3, v_1 - v_3, v_2 - v_3) \cong \mathbb{Z}[v]/(v^3)
\]
Torus Manifolds

Definition

A torus manifold $M^{2n}$ is a smooth oriented closed manifold with an effective smooth $T$-action such that $M^T \neq \emptyset$.

This implies that all fixed points are isolated.
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Multi-fans of Hattori & Masuda.
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A *torus manifold* $M^{2n}$ is a smooth oriented closed manifold with an effective smooth $T$-action such that $M^T \neq \emptyset$.

This implies that all fixed points are isolated.

Multi-fans of Hattori & Masuda.

If the $T$-action is locally standard then the quotient $Q := M/T$ is a manifold with corners.
Torus Graphs

Let $M$ be a torus manifold. For $p \in M^T$

$$T_p M \cong V_1(p) \oplus \cdots \oplus V_n(p),$$

(1)

where $V_i(p) \in \text{Hom}(T, S^1) \cong \mathbb{Z}^n$ is a complex 1-dim $T$-representation.
Torus Graphs

Let \( M \) be a torus manifold. For \( p \in M^T \)

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where \( V_i(p) \in \text{Hom}(T, S^1) \cong \mathbb{Z}^n \) is a complex 1-dim \( T \)-representation.

\[
S_M := \{ \text{2-dim submanifolds of } M \\
\text{each fixed ptwise by a codim-1 subtorus of } T \}.\]

Every \( S \in S_M \) is diffeo to a 2-sphere and contains exactly two \( T \)-fixed points.
Torus Graphs

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Every $S \in S_M$ is diffeo to a 2-sphere and contains exactly two $T$-fixed points.

Define an $n$-valent graph $\Gamma_M$ whose vertex set is $M^T$ and whose edges correspond to the 2-spheres from $S_M$. Label each oriented edge with its corresponding weight from (1).
Example $S^{2n}$

Taking the torus manifolds $S^{2n}$, where the torus action is obtained by suspending the standard coordinatewise torus action on $S^{2n-1}$, we obtain a graph with two vertices and $n$ edges between them.

The edges are labelled by the standard basis vectors \{t_1, \ldots, t_n\} of

\[ H^2 BT \cong \text{Hom}(T, S^1) \cong \mathbb{Z}^n. \]

\[ t_n \]
\[ \vdots \]
\[ t_2 \]
\[ t_1 \]

Obviously these are not toric or quasitoric manifolds for $n > 1$. 
Abstract Torus Graphs

Let
- $\Gamma$ be an $n$-valent connected graph with $n \geq 1$.
- $\mathcal{V}(\Gamma)$ denote the set of vertices.
- $\mathcal{E}(\Gamma)$ denote the set of oriented edges.
Abstract Torus Graphs

Let

- $\Gamma$ be an $n$-valent connected graph with $n \geq 1$.
- $\mathcal{V}(\Gamma)$ denote the set of vertices.
- $\mathcal{E}(\Gamma)$ denote the set of oriented edges.

For $p \in \mathcal{V}(\Gamma)$, define

$$\mathcal{E}(\Gamma)_p := \{ e \in \mathcal{E}(\Gamma) \mid i(e) = p \}.$$
Abstract Torus Graphs

Definition (Axial Function)

An axial function is a map

\[ \alpha : \mathcal{E}(\Gamma) \longrightarrow \text{Hom}(T, S^1) \cong \mathbb{Z}^n, \]

satisfying the following conditions:

1. \( \alpha(\bar{e}) = \pm \alpha(e); \)
2. elements of \( \alpha(\mathcal{E}(\Gamma)_p) \) form a basis of \( \mathbb{Z}^n; \)
3. \( \alpha(\mathcal{E}(\Gamma)_{t(e)}) \equiv \alpha(\mathcal{E}(\Gamma)_{i(e)}) \mod \alpha(e), \) for any \( e \in \mathcal{E}(\Gamma). \)
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Definition (Torus Graph)

A torus graph is a pair \((\Gamma, \alpha)\) consisting of an \(n\)-valent graph \(\Gamma\) with an axial function \(\alpha\).
\[ H^{\text{odd}} M = 0 \]

\[ H^{\text{odd}} M = 0 \implies T\text{-action is locally standard} \]

\[ \implies Q \text{ is a manifold with corners} \]
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H^{\text{odd}} M = 0 \iff T\text{-action is locally standard and } Q \text{ is face-acyclic} \\
\implies M = T \times Q/ \sim
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\[
H^{\text{odd}} M = 0 \iff H^*_T M \cong H^* M \otimes H^* BT, \text{ as } H^* BT\text{-modules} \\
\implies H^*_T M \text{ is a free } H^* BT\text{-module} \\
\implies \text{Serre SS of } M \to ET \times_T M \to BT \text{ collapses} \\
\implies i^*: H^*_T M \longrightarrow H^*_T M^T \cong \bigoplus_{p \in M^T} H^* BT \text{ is injective}
\]

How can we describe the image of \( i^* \)?
Graph Cohomology (Piecewise Polynomials)

\[ H_T^*\Gamma := \{ f : \mathcal{V}(\Gamma) \to H^*BT \mid f(i(e)) \equiv f(t(e)) \mod \alpha(e) \} \]
Graph Cohomology (Piecewise Polynomials)

\[ H_T^* \Gamma := \{ f : \mathcal{V}(\Gamma) \to H^* BT \mid f(i(e)) \equiv f(t(e)) \mod \alpha(e) \} \]

**Theorem (GKM, Masuda & Panov)**

*If \( \text{odd} M = 0 \), then \( H_T^* M \cong H_T^* \Gamma \).*
Graph Cohomology (Piecewise Polynomials)

\[ H^*_T \Gamma := \{ f : \mathcal{V}(\Gamma) \to H^*BT \mid f(i(e)) \equiv f(t(e)) \mod \alpha(e) \} \]

Theorem (GKM, Masuda & Panov)

If \( H^{odd}M = 0 \), then

\[ H^*_TM \cong H^*_T\Gamma. \]

Proof (Sketch).

All isotropy subgroups are connected. Hence the Chang-Skjelbred sequence

\[ 0 \to H^*_TM \xrightarrow{i^*} H^*_TM_0 \xrightarrow{\delta} H^{*+1}_T(M_1, M_0) \to \cdots \]

is exact with integer coefficients [Franz & Puppe '07], where \( M_0 \) and \( M_1 \) denote the set of fixed points and 1-dim orbits in \( M \) resp.
Proof (Sketch).

So

\[ H^*_T M \cong \text{Ker} \, \delta, \]

which we can rewrite as

\[ \delta : \bigoplus_{p \in M^T} H^* BT \longrightarrow \bigoplus_{e \in \mathcal{E}(\Gamma)} H^* BT_e, \]

where \( T_e := \text{Ker} \, \alpha(e) \cong T^{n-1} \), and is defined by

\[ \delta(\{f(p)\}_{p \in M^T}) = \{f(i(e))|_{H^*(BT_e)} - f(t(e))|_{H^*(BT_e)}\}_{e \in \mathcal{E}(\Gamma)}. \]

The result follows.
Thom Classes

For any $k$-dim face $F$ of $(\Gamma, \alpha)$ we define the *Thom class* of $F$ as a map

$$\tau_F : V(\Gamma) \longrightarrow H^{2(n-k)} BT$$

$$\tau_F(p) := \begin{cases} 
\prod_{i(e) = p, \ e \notin F} \alpha(e), & \text{if } p \in V(F); \\
0, & \text{otherwise.}
\end{cases}$$
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**Lemma**

$$\tau_F \in H^*_T \Gamma \quad \& \quad \tau_G \tau_H = \tau_{G \vee H} \cdot \sum_{E \in G \cap H} \tau_E.$$
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Lemma

$$\tau_F \in H^*_T \Gamma \quad \& \quad \tau_G \tau_H = \tau_{G \vee H} \cdot \sum_{E \in G \cap H} \tau_E.$$ 

Theorem (Masuda & Panov '06)

$$H^*_T \Gamma \cong \mathbb{Z}[\tau_F \mid F \text{ a face}]/\mathcal{I},$$

where $\mathcal{I} = \langle \tau_G \tau_H - \tau_{G \vee H} \cdot \sum_{E \in G \cap H} \tau_E \rangle$. 

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Toric & Quasitoric Orbifolds

*Orbifold* – locally like $\mathbb{R}^n / G$ for some finite group $G$.

Toric Orbifolds $\leftrightarrow$ comp. simplicial fans

$(\mathbb{C}^*)^n \circlearrowleft X^{2n}$ $\leftrightarrow$ $\Sigma \subseteq \mathbb{R}^n$

compact simplical toric variety

$\text{Loc. std. st. } X / T \sim \mathbb{P} \lambda : F(\mathbb{P}) \rightarrow \mathbb{Z}^n$

$\mathbb{P}$ is a simple polytope satisfies linear independence condition
**Toric & Quasitoric Orbifolds**

*Orbifold* – locally like $\mathbb{R}^n/G$ for some finite group $G$.

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\text{Toric Orbifolds} \quad \longleftrightarrow \quad \text{comp. simplicial fans} \\
(C^*)^n \circ X^{2n} \quad \longleftrightarrow \quad \Sigma \subseteq \mathbb{R}^n
\]

compact simplical toric variety

\[
\text{Quasitoric Orbifolds} \quad \longleftrightarrow \quad \text{Characteristic Pairs} \\
T^n \circ X^{2n} \quad \longleftrightarrow \quad (P, \lambda) \\
\text{loc. std. st. } X/T \cong P \quad \lambda: \mathcal{F}(P) \to \mathbb{Z}^n
\]

$P$ is a simple polytope

satisfies linear independence condition
**Weighted Projective Space**

Given a *weight vector* $\chi = (\chi_0, \ldots, \chi_n) \in \mathbb{N}^{n+1}$, define

$$\mathbb{P}(\chi) := S^{2n+1}/S^1\langle \chi \rangle,$$

where $t \cdot (z_0, \ldots, z_n) = (t^{\chi_0}z_0, \ldots, t^{\chi_n}z_n)$.

Weighted projective spaces are examples of (quasi)toric orbifolds over the simplex.
**Weighted Projective Space**

Given a weight vector $\chi = (\chi_0, \ldots, \chi_n) \in \mathbb{N}^{n+1}$, define

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where $t \cdot (z_0, \ldots, z_n) = (t^{\chi_0}z_0, \ldots, t^{\chi_n}z_n)$.

Weighted projective spaces are examples of (quasi)toric orbifolds over the simplex.

**Theorem (Kawasaki ’73)**

$$H^i_{P(\chi)} \cong \begin{cases} \mathbb{Z}, & 0 \leq i = 2j \leq 2n; \\ 0, & otherwise. \end{cases}$$

with a twisted product structure

$$\gamma_i \cup \gamma_j = \frac{l_i l_j}{l_{i+j}} \gamma_{i+j}.$$
Equivariant Cohomology

Theorem (Poddar & Sarkar ’10)

\[ H_T^*(X(\Sigma); \mathbb{Q}) \cong \mathbb{Q}[\Sigma] \]
\[ H_T^*(X(P, \lambda); \mathbb{Q}) \cong \mathbb{Q}[P] \]

What about integer coefficients?
Torus Orbifolds

Definition

A *torus orbifold* $X^{2n}$ is a smooth oriented closed orbifold with an effective smooth $T$-action such that $X^T \neq \emptyset$.

Definition (Rational Axial Function)

A *rational axial function* is a map $\alpha: E(\Gamma) \to H_2(BT; \mathbb{Q}) \cong \mathbb{Q}^n$, satisfying the following conditions:

1. For all $e \in E(\Gamma)$, there exist $r_e, r_{\bar{e}} \in \mathbb{Z}$ such that $r_e \alpha(e) = \pm r_{\bar{e}} \alpha(\bar{e}) \in H_2(BT; \mathbb{Z})$;
2. Elements of $\alpha(E(\Gamma))$ are linearly independent in $\mathbb{Q}^n$;
3. $\alpha(E(\Gamma)_t(e)) \equiv \alpha(E(\Gamma)_i(e)) \mod r_e \alpha(e)$, for any $e \in E(\Gamma)$.
Torus Orbifolds

**Definition**

A *torus orbifold* $X^{2n}$ is a smooth oriented closed orbifold with an effective smooth $T$-action such that $X^T \neq \emptyset$.

**Definition (Rational Axial Function)**

A *rational axial function* is a map

$$\alpha : \mathcal{E}(\Gamma) \rightarrow H^2(BT; \mathbb{Q}) \cong \mathbb{Q}^n,$$

satisfying the following conditions:

1. $\forall e \in \mathcal{E}(\Gamma), \exists r_e, r_\bar{e} \in \mathbb{Z}$ st. $r_e \alpha(e) = \pm r_\bar{e} \alpha(\bar{e}) \in H^2(BT; \mathbb{Z});$
2. elements of $\alpha(\mathcal{E}(\Gamma)_p)$ are linearly independent in $\mathbb{Q}^n$;
3. $\alpha(\mathcal{E}(\Gamma)_{t(e)}) \equiv \alpha(\mathcal{E}(\Gamma)_{i(e)}) \mod r_e \alpha(e)$, for any $e \in \mathcal{E}(\Gamma)$. 
Graph Cohomology

Given a *rational torus graph* \((\Gamma, \alpha)\) we define its cohomology as follows:

\[
H^*_T \Gamma := \{ f : \mathcal{V}(\Gamma) \to H^*(BT; \mathbb{Z}) \mid f(i(e)) \equiv f(t(e)) \mod r_e \alpha(e) \},
\]

\[
H^*_T(\Gamma; \mathbb{Q}) := \{ f : \mathcal{V}(\Gamma) \to H^*(BT; \mathbb{Q}) \mid f(i(e)) \equiv f(t(e)) \mod \alpha(e) \}.
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Graph Cohomology

Given a rational torus graph \((\Gamma, \alpha)\) we define its cohomology as follows:

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H_T^* \Gamma := \{ f : V(\Gamma) \to H^*(BT; \mathbb{Z}) | f(i(e)) \equiv f(t(e)) \mod r_e \alpha(e) \},
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H_T^*(\Gamma; \mathbb{Q}) := \{ f : V(\Gamma) \to H^*(BT; \mathbb{Q}) | f(i(e)) \equiv f(t(e)) \mod \alpha(e) \}.
\]

Theorem (D., Kuroki & Song)

Let \(X\) be a torus orbifold. Then

\[
H^{odd} X = 0 \implies H_T^* X \cong H_T^* \Gamma
\]

as \(H^*(BT)\)-algebras.
Graph Cohomology

Define the map

\[ \varphi: \mathbb{Q}[x_F \mid F \text{ a face}] \longrightarrow H_T^*(\Gamma; \mathbb{Q}) \]

\[ x_F \longmapsto \tau_F \]

and consider the following subring of \( \mathbb{Q}[x_F \mid F \text{ a face}] \):

\[ \mathbb{Z}\{\Gamma\} := \{ f \in \mathbb{Q}[x_F \mid F \text{ a face}] \mid \forall v \in \mathcal{V}(\Gamma), \varphi(f)|_v \in H^*(BT; \mathbb{Z}) \}. \]

This set is closed under the addition and multiplication induced from \( \mathbb{Q}[x_F \mid F \text{ a face}] \).
Graph Cohomology

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and consider the following subring of $$\mathbb{Q}[x_F \mid F \text{ a face}]:$$

$$\mathbb{Z}\{\Gamma\} := \{f \in \mathbb{Q}[x_F \mid F \text{ a face}] \mid \forall v \in \mathcal{V}(\Gamma), \varphi(f)|_v \in H^*(BT; \mathbb{Z})\}.$$  

This set is closed under the addition and multiplication induced from $$\mathbb{Q}[x_F \mid F \text{ a face}].$$

**Theorem (D., Kuroki & Song)**

$$H_T^*\Gamma \cong \mathbb{Z}\{\Gamma\}/\mathcal{I},$$

where

$$\mathcal{I} = \langle x_F x_G - x_F \lor G \sum_{E \in F \cap G} x_E \rangle.$$
Corollaries

Corollary

Let $X$ be a torus orbifold such that $H^{\text{odd}}X = 0$. Then

$$H_T^*X \cong H_T^*\Gamma \cong \mathbb{Z}\{\Gamma\}/\mathcal{I}.$$
Corollaries

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Let $X$ be a torus orbifold such that $H^{odd}X = 0$. Then

$$H^*_TX \cong H^*_T\Gamma \cong \mathbb{Z}\{\Gamma\}/\mathcal{I}.$$ 

Corollary

Let $X$ be a torus orbifold such that $H^{odd}X = 0$. Then

$$H^*X \cong \mathbb{Z}\{\Gamma\}/\mathcal{I} + \mathcal{J},$$

where $\mathcal{J}$ is given by linear relations that can be read off from the combinatorial data.
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where $\mathcal{J}$ is given by linear relations that can be read off from the combinatorial data.

This has been done for the limited case of projective toric orbifolds by Bhari, Sarkar & Song ’17.
Thank you!