Volume growth in the component of the fibered Dehn twist

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Contents

- Slow volume growth
- Liouville domain and Fibered Dehn twist
- Wrapped Floer homology
- Main Result

Definition of the slow volume growth

- Let (M^n, g) be a Riemannian manifold
- Let $\phi : M \to M$ be a compactly supported diffeomorphism, i.e. $\mathrm{supp}(\phi) = \overline{\{x \in M \mid \phi(x) \neq x\}}$ is compact.

Definition

The i-**dimensional slow volume growth of** ϕ is defined to be

$$s_i(\phi) = \sup_{\sigma \subset M:\ i\text{-cube}} \liminf_{m \to \infty} \frac{\log \mu_g(\phi^m(\sigma))}{\log m} \in [0, \infty],$$

where μ_g is the volume measure induced by a metric g.

Intuitively, this measures the complexity of a given diffeomorphism.

Examples

Examples

1. Let $\phi = id$ on M^n. Then $s_i(\phi) = 0$ for all $i = 1, 2, \cdots, n$.
2. Consider $M = S^1 \times \mathbb{R}$ and let $\tau : M \to M$ be a twisting map. Denote $L_x = \{x\} \times [-1, 1]$ the fiber at $x \in S^1$. Then $s_1(\tau) \geq 1$ and $s_2(\tau) = 0$. (In fact, τ is a **Dehn twist**.)

We are interested in a lower bound for the slow volume growth.

Crucial Idea

Alternative argument

Observe that $|\tau^m(L_x) \cap L_y| = m$ for all $y \in S^1$ with $x \neq y$.
$\Longrightarrow \mu(\tau^m(L_x)) \geq m \cdot \mu(S^1)$
$\Longrightarrow s_1(\tau) \geq 1$.

This is adaptable to a symplectic setup.

Symplectic view point

- View $\tau^m(L_x) \cap L_y$ as an intersection of two Lagrangians in $T^*S^1 \cong S^1 \times \mathbb{R}$.
- Recall that (chain level) generators of Lagrangian Floer homology $HF_*(\tau^m(L_x), L_y)$ are intersection points.
- $\dim HF_*(\tau^m(L_x), L_y)$ gives a lower bound for the number of intersection points.
- Apply the previous to obtain a lower bound for $s_1(\tau)$.

Liouville domains

Definition

A symplectic manifold (W, ω) with boundary is called a **Liouville domain** if

- There exists a 1-form λ on W such that $\omega = d\lambda$.
- Let X be the vector field $d\lambda$-dual to λ. Then X points outward along boundary ∂W.

Facts

- If (W, λ) is a Liouville domain, then $(\partial W, \xi = \ker \lambda|_{\partial W})$ is a contact manifold. (which is an odd dimensional cousin of symplectic manifolds.)
- There is a **Reeb vector field** R on ∂W, i.e. $d\alpha(R, \cdot) = 0$ and $\alpha(R) = 1$, where α is a contact form on $(\partial W, \xi)$.

Liouville domains with (P)

We need a special S^1-action on the boundary to consider the fibered Dehn twist.

Definition

We say that a Liouville domain (W, λ) satisfies **the condition** (P) if contact boundary has a periodic Reeb flow, i.e. $Flow_t^R : \partial W \to \partial W$ is periodic.

Hence, such a Reeb flow induces an S^1-action on ∂W.

Definition of the fibered Dehn twist

Definition

Let (W, λ) be a Liouville domain with (P). A symplectomorphism $\tau : W \to W$ is called the **fibered Dehn twist on** W if it is constructed as following picture:

The Reeb condition on the S^1-action guarantees that τ is in fact a symplectomorphism.

Examples of Liouville domains with (P)

Examples

1. $W := (D^{2n}, \omega_0)$ is the standard $2n$-ball and $\partial W = S^{2n-1} \to \mathbb{C}P^{n-1}$ is the Hopf fibration .
2. Consider
   - (M^{2n}, ω) : closed symplectic manifold with $[\omega] \in H^2(M; \mathbb{Z})$.
   - $Q^{2n-2} \subset M$: symplectic hypersurface PD to $k[\omega]$, $k \geq 1$.

   Then, $W := M \setminus \nu_M(Q)$ is the complement of a hypersurface and ∂W is a S^1-bundle over Q with $c_1 = k[\omega|_Q]$.
3. Let $A_k = \{z_0^{k+1} + z_1^2 + \cdots + z_n^2 = 1\} \subset \mathbb{C}^{n+1}$ be the A_k-singularity. Then, $W := A_k \cap Ball_r^{2n+2}$ and ∂W is the link of A_k-singularity.

Question

Is there an uniform lower bound for the slow volume growth of symplectomorphisms on a Liouville domain (W, λ) with (P)?

- We answer this question on the component of the fibered Dehn twist τ in $Symp(W, \partial W)$.
- More precisely, we examine a lower bound for s_n on symplectomorphisms which are symplectically isotopic to τ^k.

Uniform lower bounds are obtained in the following cases:

- closed symplectic manifolds with $\pi_2 = 0$ by [Polterovich 02'].
- Dehn-Seidel twists by [Frauenfelder and Schlenk '05].

Wrapped Floer homology $HW_*(L; W)$

Let (W, λ) be a Liouville domain and $L \subset W$ an admissible Lagrangian.

- By attaching suitable infinite cone, we obtain completions $(\hat{W}, \hat{\lambda})$ and $\hat{L} \subset \hat{W}$.
- $H : \hat{W} \to \mathbb{R}$: an admissible Hamiltonian
- J : a $d\hat{\lambda}$-compatible almost complex structure on $\hat{W}$.
- $\Omega(\hat{W}, \hat{L}) = \{x : [0,1] \to \hat{W} \mid x(0), x(1) \in \hat{L}\}$: a path space from $\hat{L}$ to itself.
- the action functional of H :

$$\begin{aligned} \mathcal{A}_H : \Omega(\hat{W}, \hat{L}) &\longrightarrow \mathbb{R} \\ x &\longmapsto -\int_0^1 x^*\hat{\lambda} + \int_0^1 H(x(t))dt \end{aligned}$$

Then, $Crit(\mathcal{A}_H)$ corresponds to the set of Hamiltonian chords. Recall that $x \in \Omega(\hat{W}, \hat{L})$ is a Hamiltonian chord if $\dot{x} = X_H \circ x$.

crit pt of $\mathcal{A}_H \leftrightarrow$ Ham chord $\to$ intersection pt of $\hat{L} \cap Flow_1^{X_H}(\hat{L})$

Wrapped Floer homology $HW_*(L; W)$

- (Chain group)

$$CW_i(H) = \bigoplus_{\substack{x \text{ : Ham chord} \\ \text{s.t. } \mu(x)=i}} \mathbb{Z}_2\langle x \rangle$$

  is a $\mathbb{Z}_2$-vector space generated by Hamiltonian chords with Maslov index $i \in \mathbb{Z}$.

- (Boundary map)

$$\begin{aligned} \partial : CW_i(H) &\longrightarrow CW_{i-1}(H) \\ x &\longmapsto \sum_{\mu(x)-\mu(y)=1} \#_{\mathbb{Z}_2}\mathcal{M}(x,y) \cdot y, \end{aligned}$$

  where $\mathcal{M}(x,y)$ is the moduli space of Floer strips from x to y.

- For a generic (H,J), $HW_*(H,J) := H_*(CW_*(H), \partial)$ is the wrapped Floer homology of (H,J).

Wrapped Floer homology $HW_*(L; W)$

Definition

The direct limit $HW_*(L; W) := \varinjlim_{slope(H)\to\infty} HW_*(H, J)$ is called the wrapped Floer homology of (W, L).

- It does not depend on the choice of (H, J).
- It is invariant up to a Hamiltonian isotopy.
- There are two types of generators on chain levels:
  1. Hamiltonian chords inside W.
  2. Hamiltonian chords on the infinite cone $[1, \infty) \times \partial W$.
- If (W, λ) is a Liouville domain with (P), then generators of type two correspond to intersection points of $L \cap \tau(L), L \cap \tau^2(L), L \cap \tau^3(L), \cdots$.

Hence, the dimension of $HW_*(L; W)$ gives a lower bound for intersection points of $L \cap \tau(L), L \cap \tau^2(L), L \cap \tau^3(L), \cdots$.

Main Result

Main theorem A

Let (W^{2n}, λ) be a Liouville domain with (P) and $H^1_c(W; \mathbb{R}) = 0$. Suppose that there is an admissible Lagrangian $L \subset W$ such that

- L is diffeomorphic to D^n.
- $\liminf_{c \to \infty} \dfrac{\dim HW_*^{<c}(L; W)}{c} > 0.$

Then, for any $\phi \in Symp(W, \partial W)$ with $\phi \overset{\simeq}{_{sympl}} \tau^k$ for some $k \geq 1$, we have $s_n(\phi) \geq 1$.

Main theorem B

By the Morse-Bott spectral sequence for $HW_*(L; W)$, we have various examples applied to theorem A.

Real symplectic manifolds and real Lagrangians

Let $\rho : M \to M$ be an anti-symplectic involution on a symplectic manifold (M, ω), i.e. $\rho^2 = id$ and $\rho^*\omega = -\omega$.
Then, $L := Fix(\rho) \subset M$ is a (real) Lagrangian if $L \neq \varnothing$.

Examples

1. $W := (D^{2n}, \omega_0)$: the standard $2n$-ball and the Hopf fibration $\partial W = S^{2n-1} \to \mathbb{C}P^{n-1}$ and $L = D^n \Longrightarrow HW_*(L; W) = 0$.
2. Consider
   - (M^{2n}, ω, ρ) : closed, monotone, real, symplectic manifold with $[\omega] \in H^2(M; \mathbb{Z})$.
   - $Q^{2n-2} \subset M$: ρ-invariant symplectic hypersurface PD to $k[\omega]$, $k \geq 1$.

   Then, complement $W := M \setminus \nu_M(Q)$ and $L = Fix(\rho|_W)$.
3. Let $A_k = \{z_0^{k+1} + z_1^2 + \cdots + z_n^2 = 1\} \subset \mathbb{C}^{n+1}$ be the A_k-singularity. Then, $W := A_k \cap Ball_r^{2n+2}$ and $L = \mathbb{R}^{n+1} \cap W$.

Upshot

Fortunately, we did not use a nuclear bomb to kill mosquito!

Genuine symplectic phenomenon

There are examples satisfying

- Main theorem.
- τ^k is smoothly isotopic to the identity map id for some $k \geq 1$.

Recall that $s_n(id) = 0$.

- Smooth isotopies do not guarantee an uniform lower bound for s_n.
- But, symplectic isotopies (from τ^k) guarantee the uniform lower bound for s_n.

Thank you for your attention. Again, I appreciate organizers.