Volume growth in the component of the fibered Dehn twist

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Definition of the slow volume growth

- Let \((M^n, g)\) be a Riemannian manifold.
- Let \(\phi : M \rightarrow M\) be a compactly supported diffeomorphism, i.e. \(\text{supp}(\phi) = \{x \in M \mid \phi(x) \neq x\}\) is compact.

**Definition**

The \(i\)-dimensional slow volume growth of \(\phi\) is defined to be

\[
s_i(\phi) = \sup_{\sigma \subset M: \ i\text{-cube}} \liminf_{m \rightarrow \infty} \frac{\log \mu_g(\phi^m(\sigma))}{\log m} \in [0, \infty],
\]

where \(\mu_g\) is the volume measure induced by a metric \(g\).

Intuitively, this measures the complexity of a given diffeomorphism.
Examples

1. Let $\phi = id$ on $M^n$. Then $s_i(\phi) = 0$ for all $i = 1, 2, \cdots, n$.  

2. Consider $M = S^1 \times \mathbb{R}$ and let $\tau : M \to M$ be a twisting map. Denote $L_x = \{x\} \times [-1, 1]$ the fiber at $x \in S^1$. Then $s_1(\tau) \geq 1$ and $s_2(\tau) = 0$. (In fact, $\tau$ is a **Dehn twist**.)

We are interested in a lower bound for the slow volume growth.
Observe that $|\tau^m(L_x) \cap L_y| = m$ for all $y \in S^1$ with $x \neq y$.

$$\Rightarrow \quad \mu(\tau^m(L_x)) \geq m \cdot \mu(S^1)$$

$$\Rightarrow \quad s_1(\tau) \geq 1.$$

This is adaptable to a symplectic setup.

**Symplectic view point**

- View $\tau^m(L_x) \cap L_y$ as an intersection of two Lagrangians in $T^* S^1 \cong S^1 \times \mathbb{R}$.

- Recall that (chain level) generators of Lagrangian Floer homology $HF_* (\tau^m(L_x), L_y)$ are intersection points.

- $\dim HF_* (\tau^m(L_x), L_y)$ gives a lower bound for the number of intersection points.

- Apply the previous to obtain a lower bound for $s_1(\tau)$. 
Liouville domains

**Definition**

A symplectic manifold \((W, \omega)\) with boundary is called a **Liouville domain** if

- There exists a 1-form \(\lambda\) on \(W\) such that \(\omega = d\lambda\).
- Let \(X\) be the vector field \(d\lambda\)-dual to \(\lambda\). Then \(X\) points outward along boundary \(\partial W\).

**Facts**

- If \((W, \lambda)\) is a Liouville domain, then \((\partial W, \xi = \ker \lambda|_{\partial W})\) is a contact manifold. (which is an odd dimensional cousin of symplectic manifolds.)
- There is a **Reeb vector field** \(R\) on \(\partial W\), i.e. \(d\alpha(R, \cdot) = 0\) and \(\alpha(R) = 1\), where \(\alpha\) is a contact form on \((\partial W, \xi)\).
Liouville domains with \((P)\)

We need a special \(S^1\)-action on the boundary to consider the fibered Dehn twist.

**Definition**

We say that a Liouville domain \((W, \lambda)\) satisfies the condition \((P)\) if contact boundary has a periodic Reeb flow, i.e. 
\[
\text{Flow}^R_t : \partial W \to \partial W \text{ is periodic.}
\]

Hence, such a Reeb flow induces an \(S^1\)-action on \(\partial W\).
Definition

Let \((W, \lambda)\) be a Liouville domain with \((P)\). A symplectomorphism \(\tau : W \to W\) is called the **fibered Dehn twist on** \(W\) if it is constructed as following picture:

The Reeb condition on the \(S^1\)-action guarantees that \(\tau\) is in fact a symplectomorphism.
Examples of Liouville domains with \((P)\)

**Examples**

1. \(W := (D^{2n}, \omega_0)\) is the standard \(2n\)-ball and \(\partial W = S^{2n-1} \to \mathbb{C}P^{n-1}\) is the Hopf fibration.

2. Consider \((M^{2n}, \omega)\) : closed symplectic manifold with \([\omega] \in H^2(M; \mathbb{Z})\).
   - \(Q^{2n-2} \subset M\) : symplectic hypersurface PD to \(k[\omega], k \geq 1\).

   Then, \(W := M \setminus \nu_M(Q)\) is the complement of a hypersurface and \(\partial W\) is a \(S^1\)-bundle over \(Q\) with \(c_1 = k[\omega|_Q]\).

3. Let \(A_k = \{z_0^{k+1} + z_1^2 + \cdots + z_n^2 = 1\} \subset \mathbb{C}^{n+1}\) be the \(A_k\)-singularity. Then, \(W := A_k \cap Ball^{2n+2}_r\) and \(\partial W\) is the link of \(A_k\)-singularity.
Question

Is there an uniform lower bound for the slow volume growth of symplectomorphisms on a Liouville domain $(W, \lambda)$ with $(P)$?

- We answer this question on the component of the fibered Dehn twist $\tau$ in $\text{Symp}(W, \partial W)$.
- More precisely, we examine a lower bound for $s_n$ on symplectomorphisms which are symplectically isotopic to $\tau^k$.

Uniform lower bounds are obtained in the following cases:

- closed symplectic manifolds with $\pi_2 = 0$ by [Polterovich 02’].
- Dehn-Seidel twists by [Frauenfelder and Schlenk ’05].
Wrapped Floer homology $HW_*(L; W)$

Let $(W, \lambda)$ be a Liouville domain and $L \subset W$ an admissible Lagrangian.

- By attaching suitable infinite cone, we obtain completions $(\hat{W}, \hat{\lambda})$ and $\hat{L} \subset \hat{W}$.
- $H : \hat{W} \to \mathbb{R}$: an admissible Hamiltonian
- $J : a \ d\hat{\lambda}$-compatible almost complex structure on $\hat{W}$.
- $\Omega(\hat{W}, \hat{L}) = \{x : [0, 1] \to \hat{W} \mid x(0), x(1) \in \hat{L}\}$: a path space from $\hat{L}$ to itself.
- the action functional of $H$:

$$A_H : \Omega(\hat{W}, \hat{L}) \to \mathbb{R}$$

$$x \mapsto -\int_0^1 x^* \hat{\lambda} + \int_0^1 H(x(t)) dt$$

Then, $\text{Crit}(A_H)$ corresponds to the set of Hamiltonian chords. Recall that $x \in \Omega(\hat{W}, \hat{L})$ is a Hamiltonian chord if $\dot{x} = X_H \circ x$.

Crit pt of $A_H \leftrightarrow$ Ham chord $\to$ intersection pt of $\hat{L} \cap \text{Flow}^X_1(\hat{L})$
Wrapped Floer homology $HW_*(L; W)$

- (Chain group)

$$CW_i(H) = \bigoplus_{x : \text{Ham chord s.t. } \mu(x) = i} \mathbb{Z}_2 \langle x \rangle$$

is a $\mathbb{Z}_2$-vector space generated by Hamiltonian chords with Maslov index $i \in \mathbb{Z}$.

- (Boundary map)

$$\partial : CW_i(H) \longrightarrow CW_{i-1}(H)$$

$$x \longmapsto \sum_{\mu(x) - \mu(y) = 1} \#\mathcal{M}(x, y) \cdot y,$$

where $\mathcal{M}(x, y)$ is the moduli space of Floer strips from $x$ to $y$.

- For a generic $(H, J)$, $HW_*(H, J) := H_*(CW_*(H), \partial)$ is the wrapped Floer homology of $(H, J)$. 
**Definition**

The direct limit \( \text{HW}_*(L; W) := \lim_{\text{slope}(H) \to \infty} \text{HW}_*(H, J) \) is called the wrapped Floer homology of \((W, L)\).

- It does not depend on the choice of \((H, J)\).
- It is invariant up to a Hamiltonian isotopy.
- There are two types of generators on chain levels:
  1. Hamiltonian chords inside \(W\).
  2. Hamiltonian chords on the infinite cone \([1, \infty) \times \partial W\).
- If \((W, \lambda)\) is a Liouville domain with \((P)\), then generators of type two correspond to intersection points of \(L \cap \tau(L), L \cap \tau^2(L), L \cap \tau^3(L), \ldots\).

Hence, the dimension of \( \text{HW}_*(L; W) \) gives a lower bound for intersection points of \(L \cap \tau(L), L \cap \tau^2(L), L \cap \tau^3(L), \ldots\).
Main Result

Main theorem A

Let \((W^{2n}, \lambda)\) be a Liouville domain with \((P)\) and \(H^1_c(W; \mathbb{R}) = 0\). Suppose that there is an admissible Lagrangian \(L \subset W\) such that

- \(L\) is diffeomorphic to \(D^n\).
- \(\liminf_{c \to \infty} \frac{\dim HW^< c(L; W)}{c} > 0\).

Then, for any \(\phi \in \text{Symp}(W, \partial W)\) with \(\phi \sim_{\text{sympl}} \tau^k\) for some \(k \geq 1\), we have \(s_n(\phi) \geq 1\).

Main theorem B

By the Morse-Bott spectral sequence for \(HW_*(L; W)\), we have various examples applied to theorem A.
Let $\rho : M \to M$ be an anti-symplectic involution on a symplectic manifold $(M, \omega)$, i.e. $\rho^2 = id$ and $\rho^* \omega = -\omega$. Then, $L := Fix(\rho) \subset M$ is a (real) Lagrangian if $L \neq \emptyset$.

### Examples

1. $W := (D^{2n}, \omega_0)$: the standard $2n$-ball and the Hopf fibration $\partial W = S^{2n-1} \to \mathbb{C}P^{n-1}$ and $L = D^n \implies HW_*(L; W) = 0$.

2. Consider
   - $(M^{2n}, \omega, \rho)$: closed, monotone, real, symplectic manifold with $[\omega] \in H^2(M; \mathbb{Z})$.
   - $Q^{2n-2} \subset M$: $\rho$-invariant symplectic hypersurface PD to $k[\omega]$, $k \geq 1$.
   Then, complement $W := M \setminus \nu_M(Q)$ and $L = Fix(\rho|_W)$.

3. Let $A_k = \{z_0^{k+1} + z_1^2 + \cdots + z_n^2 = 1\} \subset \mathbb{C}^{n+1}$ be the $A_k$-singularity. Then, $W := A_k \cap Ball^{2n+2}_r$ and $L = \mathbb{R}^{n+1} \cap W$. 

Joontae Kim

Volume growth in the component of the fibered Dehn twist
Fortunately, we did not use a nuclear bomb to kill mosquito!

Genuine symplectic phenomenon

There are examples satisfying
- Main theorem.
- $\tau^k$ is smoothly isotopic to the identity map $id$ for some $k \geq 1$.

Recall that $s_n(id) = 0$.

- Smooth isotopies do not guarantee an uniform lower bound for $s_n$.
- But, symplectic isotopies (from $\tau^k$) guarantee the uniform lower bound for $s_n$. 
Thank you for your attention. Again, I appreciate organizers.