Wedge Operation on Toric Orbifolds

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Setting and Question

- $T = (S^1)^k$: a k-dimensional torus.
- X: a T-space.

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- X: a T-space.

(Probably) we want to calculate...

$$H^*(X),\ H_T^*(X),\ K^*(X),\ K_T^*(X),\ MU^*(X),\ MU_T^*(X), \ldots$$

Some tools

- Chang-Skjelbred sequence : $H_T^*(X)$
- Eilenberg–Moore spectral sequence : $H^*(X)$
- Atiyah–Hirzebruch spectral sequence : $K(X),\ MU(X)$
- Generalized GKM theory : $K_T^*(X),\ MU_T^*(X)$.
- $\vdots$

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Remark

If we assume $H^{odd}(X) = 0$, then the computation becomes much easier.

Many "nice" families of spaces satisfy $H^{odd}(X) = 0$

- $G \curvearrowright M$
  cpt Hamiltonian G-manifold with $|M^G| < \infty$.

- G/K, where
  G: cpt semisimple Lie group, K contains the maximal torus.

- cpt toric origami with acyclic origami template and coörientable folding hypersurface. (inspired from Zeng's talk..)

- $\vdots$

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- Quasitoric manifolds, Smooth toric varieties.

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- Quasitoric manifolds, Smooth toric varieties.

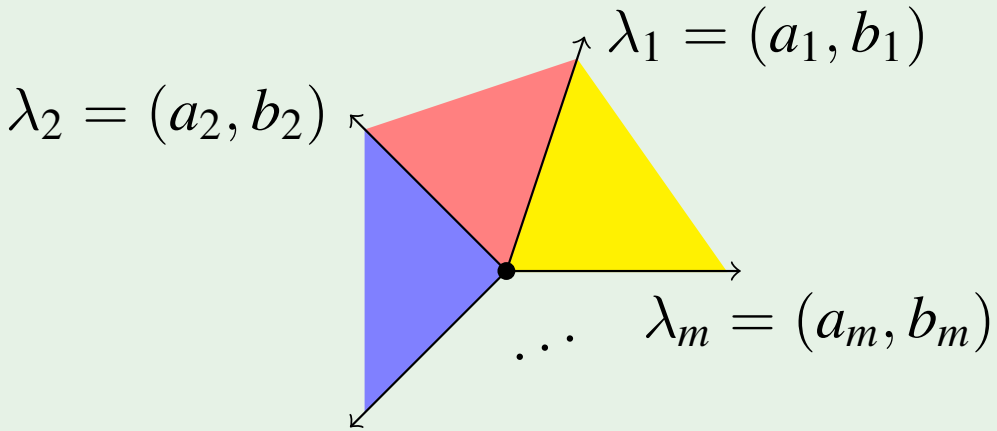
Remark

However, in general,

$$H^{odd}(\textit{toric orbifold / singular toric variety}) \neq 0.$$

Examples of toric orbifolds

Example (4-dimsnsional toric variety)



$$H^i(X(P,\lambda);\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0,\ 4 \\ 0 & i = 2 \\ \mathbb{Z}^{m-2} & i = 2 \\ \mathbb{Z}/p\mathbb{Z} & i = 3 \end{cases}$$

Question

For toric orbifolds/singular toric varieties,
are there any necessary or sufficient conditions for $H^{odd}(X) = 0$?

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are there any necessary or sufficient conditions for $H^{odd}(X) = 0$?

Partial answer

[Kuwata, Masuda, Zeng arXiv:1604.03138] Torsions in the cohomology of torus orbifolds.

- The complete answer for 4-dimensional torus orbifolds.
- A necessary condition for arbitrary dimensional torus orbifolds.

quasitoric orbifold

Ingredients: $(P,\ \lambda\colon \mathcal{F}(P) \to \mathbb{Z}^n)$

- P: n-dimensional simple polytope.
- $\mathcal{F}(P)$: set of facets of P.
- $\lambda\colon \mathcal{F}(P) \to \mathbb{Z}^n$ satisfies...
  1. $\det[\lambda(F_{i_1}) \mid \cdots \mid \lambda(F_{i_n})] \neq 0$ whenever $\bigcap_{j=1}^{n} F_{i_j} = v \in V(Q)$,
  2. $\lambda(F_i)$'s are primitive,
  3. imλ spans $\mathbb{Z}^n$.

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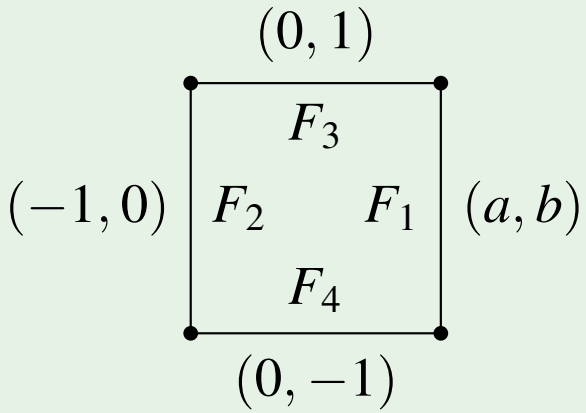
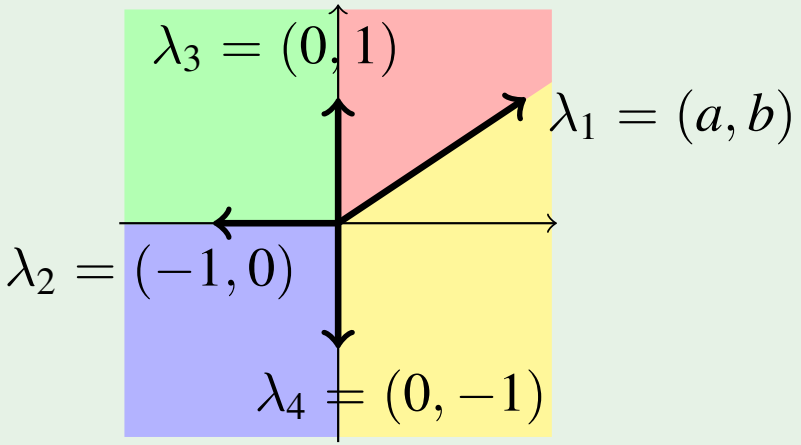
Example





quasitoric orbifolds

$$X(P,\lambda) := (T^n \times P)/\sim,$$

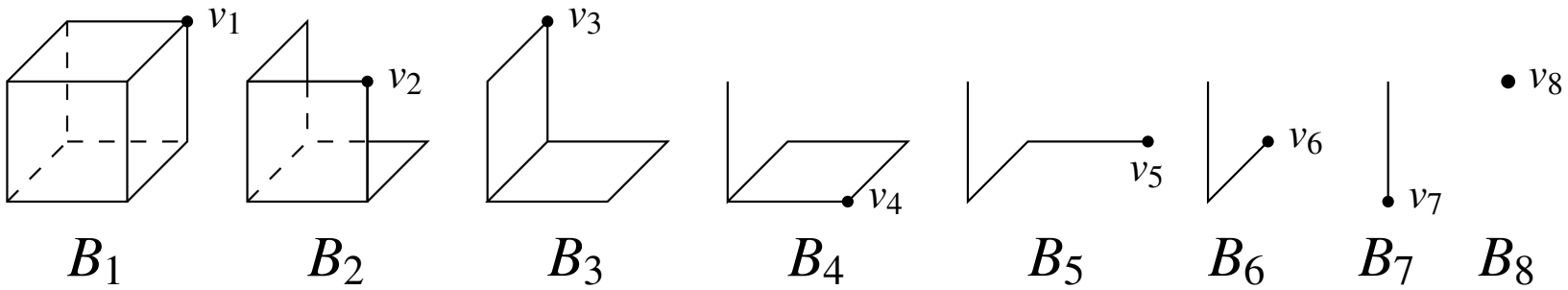
where $[t,x] \sim [s,y]$ if and only if $x = y$ and $t^{-1}s \in T_{E(x)}$.

Remark

- $T^n \curvearrowright X(P,\lambda)$ *by the multiplication on the first factor.*
- $\pi : X(P,\lambda) \to P$ *is the orbit map.*
- *For each face* E *of* P, $\pi^{-1}(E) = X(E,\lambda_E)$.

A retraction sequence

A retraction sequence is, for instance,



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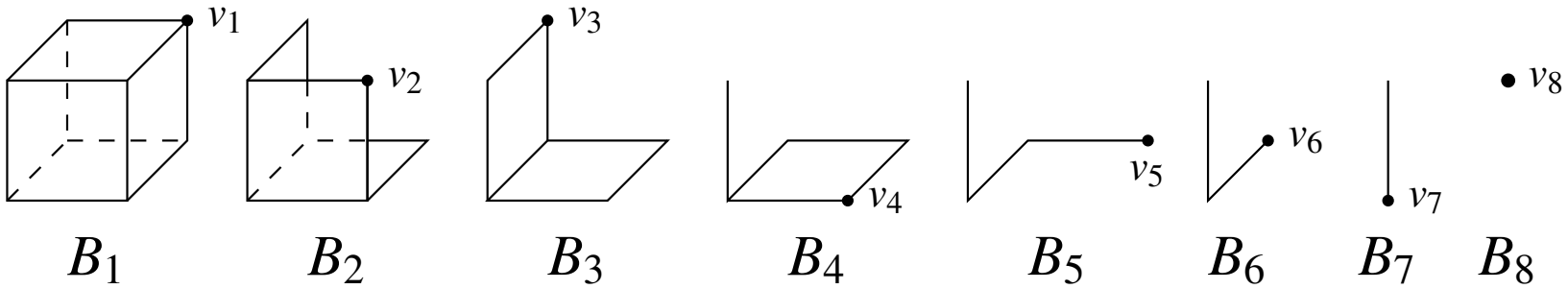
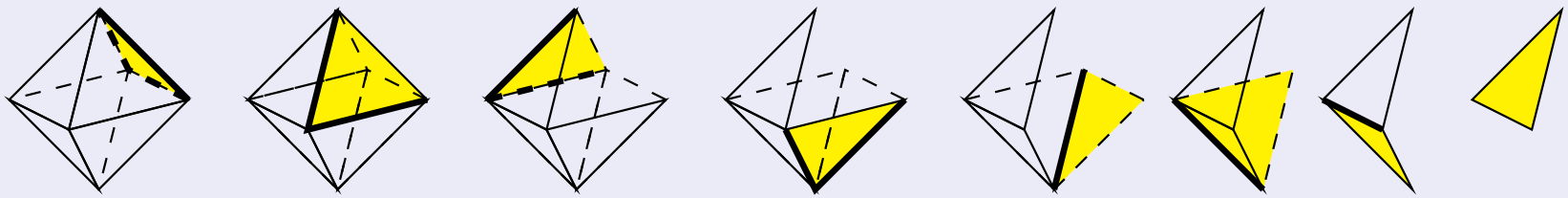
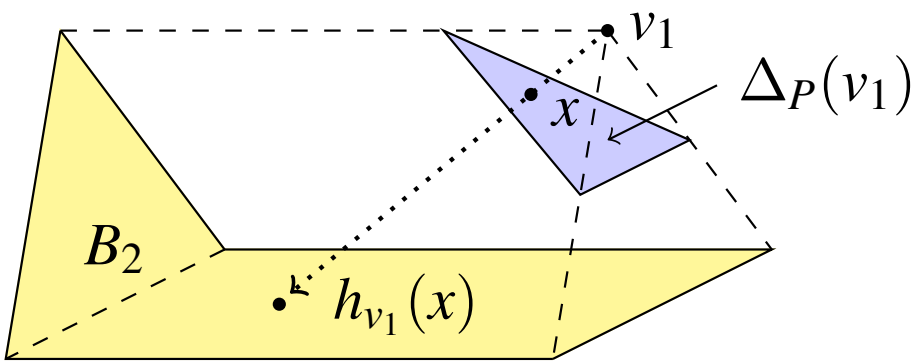


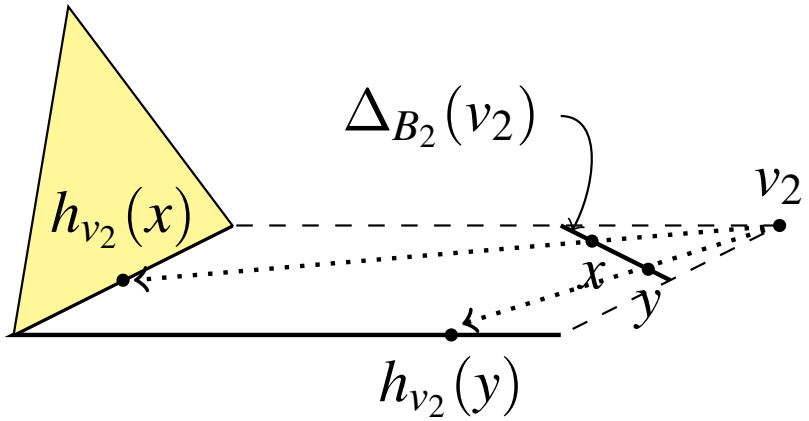
Remark

In terms of simplicial complex, (special case of) shelling.



Why retraction..?





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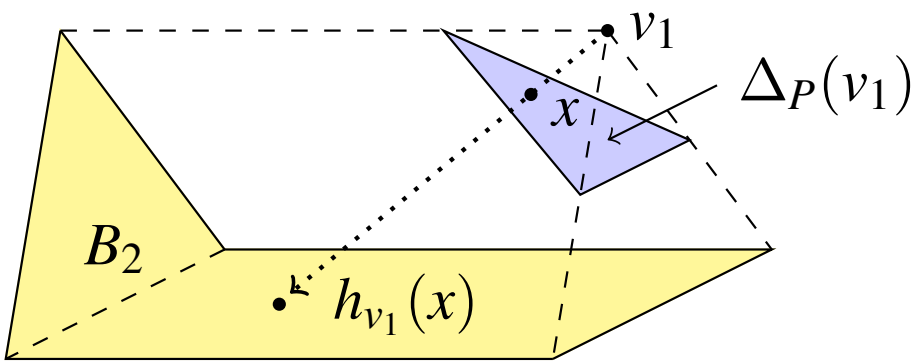

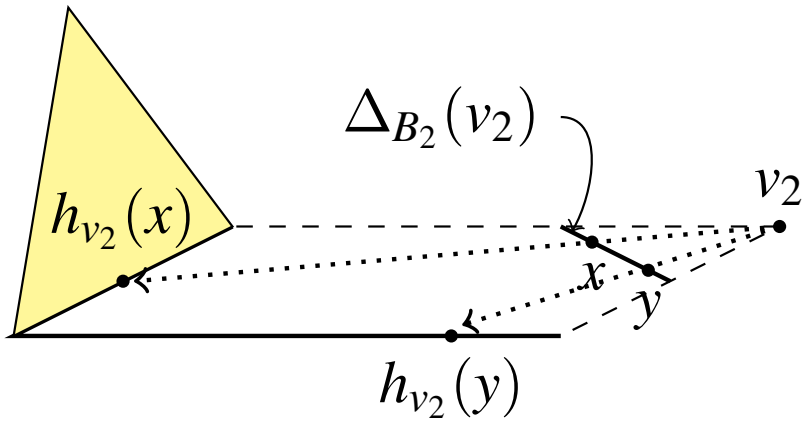

- $\pi^{-1}(P) = X(P, \lambda) =: X_P$
- $\pi^{-1}(B_2) = \bigcup_{E\text{: face of } B_2} \pi^{-1}(E)$, where $\pi^{-1}(E) = X(E, \lambda_E)$.
- $\pi^{-1}(\Delta_P(v)) = (\Delta_P \times T^n)/\sim \cong S^{2n-1}/G_P(v)$,
  where $|G_P(v)| = |\det \Lambda_v|$

How are they related?

Proposition

The composition

$$\pi^{-1}(\Delta_P(v)) \xrightarrow{f} \bigcup_{E\text{: face of } B_2} \pi^{-1}(E) \hookrightarrow X(P,\lambda)$$

is a cofiber sequence. i.e., $X(P,\lambda) \simeq c(f)$

Corollary

$$\begin{aligned} H_*(X(P,\lambda), \pi^{-1}(B_2)) &\cong H_*(C(L(\Delta_P(v), \xi_v)), \pi^{-1}(B_2)) \\ &\cong \tilde{H}_{*-1}(\pi^{-1}(\Delta_P(v))). \end{aligned}$$

Since $\pi^{-1}(E)$ is another toric orbifold...

Proposition

For each face $E \subset B_2$, the composition

$$\pi^{-1}(\Delta_E(v)) \xrightarrow{f_E} \bigcup_{F:\ \text{face of } B_3} \pi^{-1}(F) \hookrightarrow \pi^{-1}(B_2)$$

is a cofiber sequence. i.e., $\pi^{-1}(B_2) \simeq c(f_E)$

Corollary

$$\begin{aligned} H_*(\pi^{-1}(B_2), \pi^{-1}(B_3)) &\cong H_*(C(L(\Delta_E(b_2), \xi_v)), \pi^{-1}(B_3)) \\ &\cong \tilde{H}_{*-1}(\pi^{-1}(\Delta_E(b_2))). \end{aligned}$$

The long exact sequence of pair $(X, \pi^{-1}(B_2))$

$$\to H_{j+1}(X, \pi^{-1}(B_2)) \to H_j(\pi^{-1}(B_2)) \to H_j(X) \longrightarrow H_j(X, \pi^{-1}(B_2)) \longrightarrow$$

$$\begin{array}{ccc} \| & & \| \\ \tilde{H}_j(\pi^{-1}(\Delta_P(v))) & & \tilde{H}_{j-1}(\pi^{-1}(\Delta_P(v))) \\ \| & & \| \\ |G_v|\text{-torsion} & & |G_v|\text{-torsion} \end{array}$$

Main Theorem: A sufficient condition for $H^{odd} = 0$.

Theorem

For $X(P, \lambda)$, if

$$\gcd\{|G_E(v)| : v \in FV(B)\} = 1,$$

for each $B \in \mathfrak{B}(Q)$ with $\dim B \geq 1$, then, $H_*(X)$ is torsion free and concentrated in even degrees.

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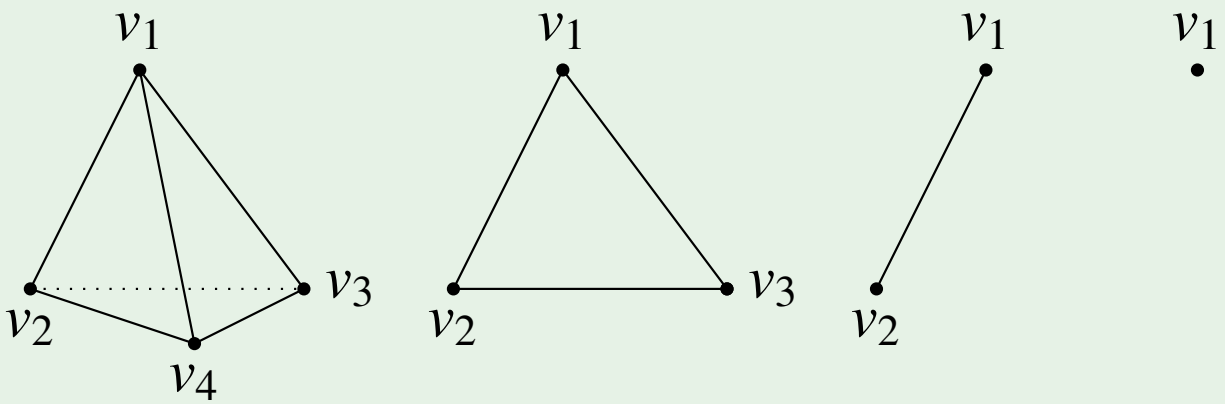
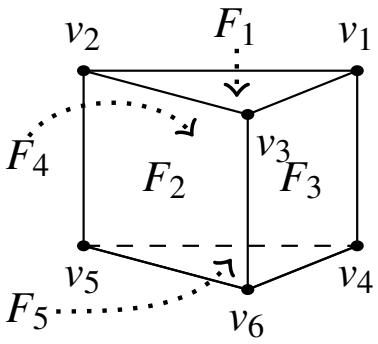
Example (Weighted projective space)

$\Delta^n \quad \Delta^{n-1} \quad \cdots$



Example: Which is not a weighted projective space



$$\begin{array}{cccccc} & \lambda(F_1) & \lambda(F_2) & \lambda(F_3) & \lambda(F_4) & \lambda(F_5) \\ \left[\begin{array}{c} \\ \\ \end{array}\right. & \begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array} & \begin{array}{c} 1 \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 1 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \end{array} & \begin{array}{c} q_1 \\ q_2 \\ q_3 \end{array} \left.\begin{array}{c} \\ \\ \end{array}\right] \end{array}$$

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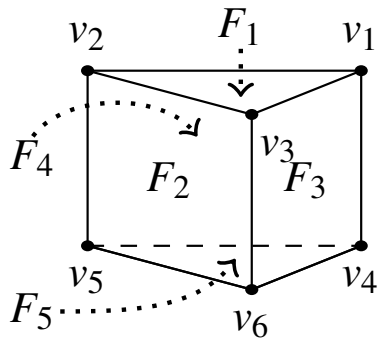

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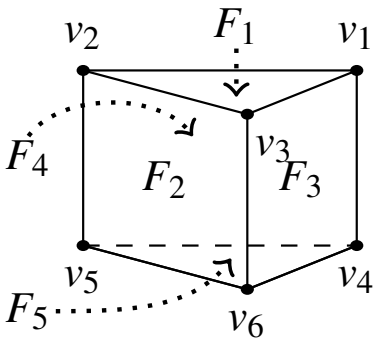
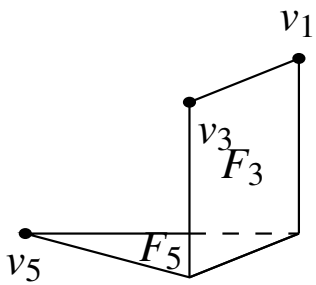
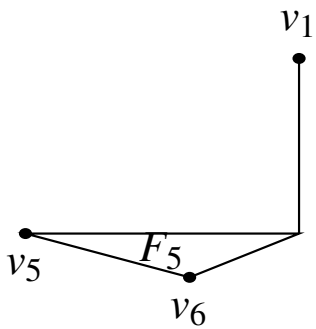


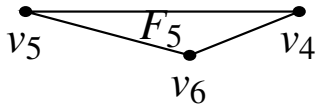
$Q = B_1$



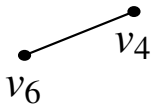
B_2



B_3



B_4

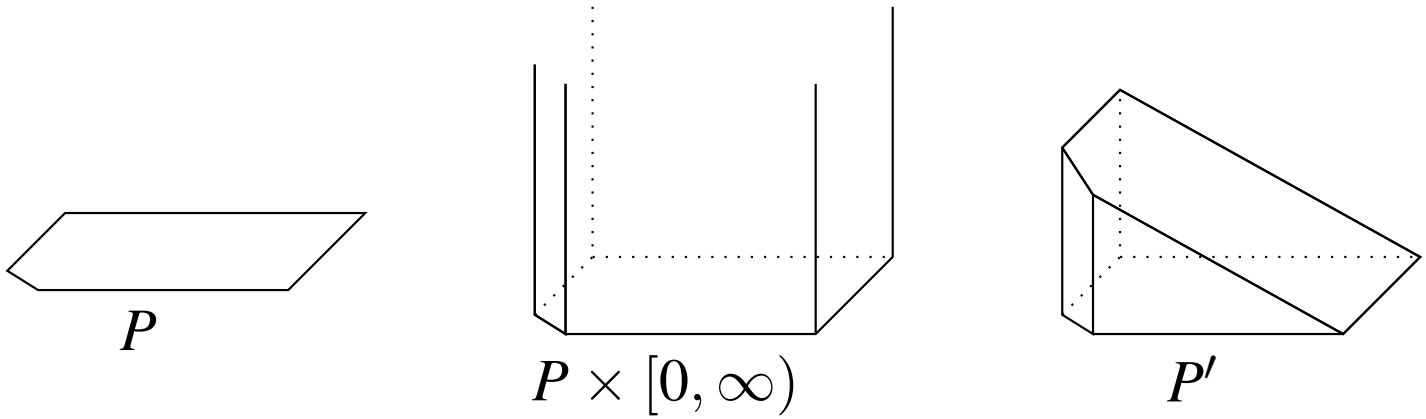


B_5

B_6

Wedge operation: $(P,\lambda) \overset{Wedge}{\rightsquigarrow} (P',\lambda') \overset{Wedge}{\rightsquigarrow} \cdots$

1. $P \rightsquigarrow P' := Wedge_{F_i} P$



- $\mathcal{F}(P') = \{F'_1, \ldots, F'_{i-1}, P^+, P^-, F'_{i+1}, \ldots, F'_m\}$.

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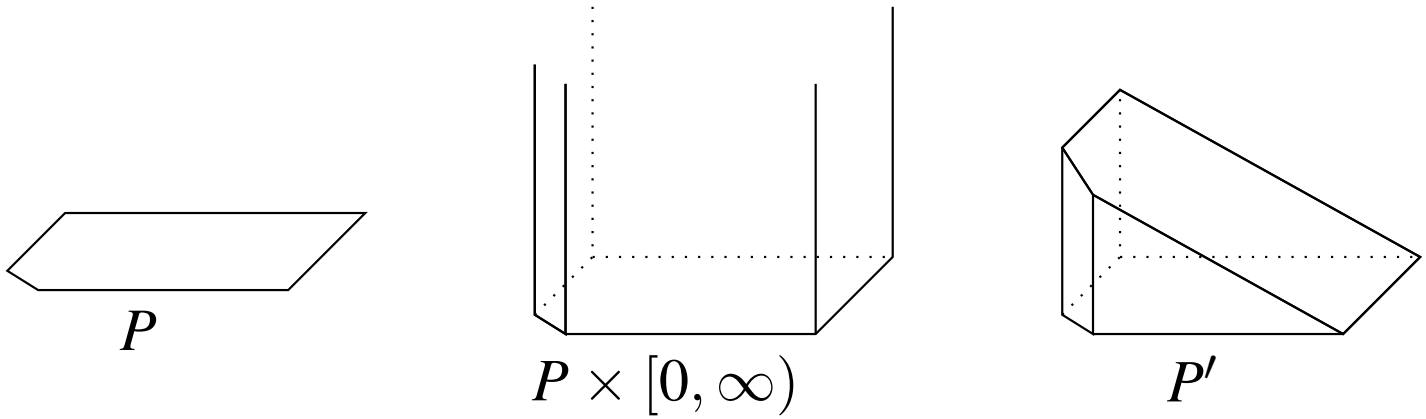

   - $\mathcal{F}(P') = \{F'_1, \ldots, F'_{i-1}, P^+, P^-, F'_{i+1}, \ldots, F'_m\}$.

2. $\lambda \rightsquigarrow \lambda'$
   - $\lambda'(F'_j) := (0, \lambda(F_j))$, $j \neq i$,
   - $\lambda'(P^+) := (-1, \lambda(F_i))$,
   - $\lambda'(P^-) := (1, 0, \ldots, 0)$.

Examples

- $(\Delta^1, [1 \ -1]) \rightsquigarrow \left(\Delta^2, \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}\right) \rightsquigarrow \left(\Delta^3, \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}\right) \rightsquigarrow \ldots$
- A Bott manifold $\rightsquigarrow$ A generalized Bott manifold
- $\vdots$

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Why "Wedge"?

- A natural way to produce a new manifold/orbifold with "similar" property.
- (Together with the theory of “polyhedral product”)
  A toric space with less combinatorics!
- $\vdots$

Example: Weighted projective spaces

- $P = \Delta^n$
- $\lambda\colon \mathcal{F}(\Delta^n) := \{F_1, \ldots, F_{n+1}\} \to \mathbb{Z}^n$ with

$$\sum_{i=1}^{n+1} \chi_i \lambda(F_i) = \mathbf{0}, \text{ for some } \chi := (\chi_1, \ldots, \chi_{n+1}) \in \mathbb{N}^{n+1}.$$

- $X(P, \lambda) \cong \mathbb{C}P^n_\chi$.

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- $P' = \Delta^{n+1}$
- $\lambda'\colon \mathcal{F}(\Delta^{n+1}) \to \mathbb{Z}^{n+1}$ satisfies

$$\sum_{j=1}^{i-1} \chi_i \lambda'(F'_j) + \chi_i \lambda'(P^-) + \chi_i \lambda'(P^+) + \sum_{j=i+1}^{n+1} \chi_j \lambda'(F'_j) = \mathbf{0}.$$

- $X(P', \lambda') \cong \mathbb{C}P^{n+1}_{(\chi_1, \ldots, \chi_i, \chi_i, \ldots, \chi_{n+1})}$.

Singularities..

Proposition

For arbitrary (P, λ),

$$\{|G_P(v)| \;\big|\; v \in V(P)\} = \{|G_P(v')| \;\big|\; v \in V(P')\}.$$

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Example

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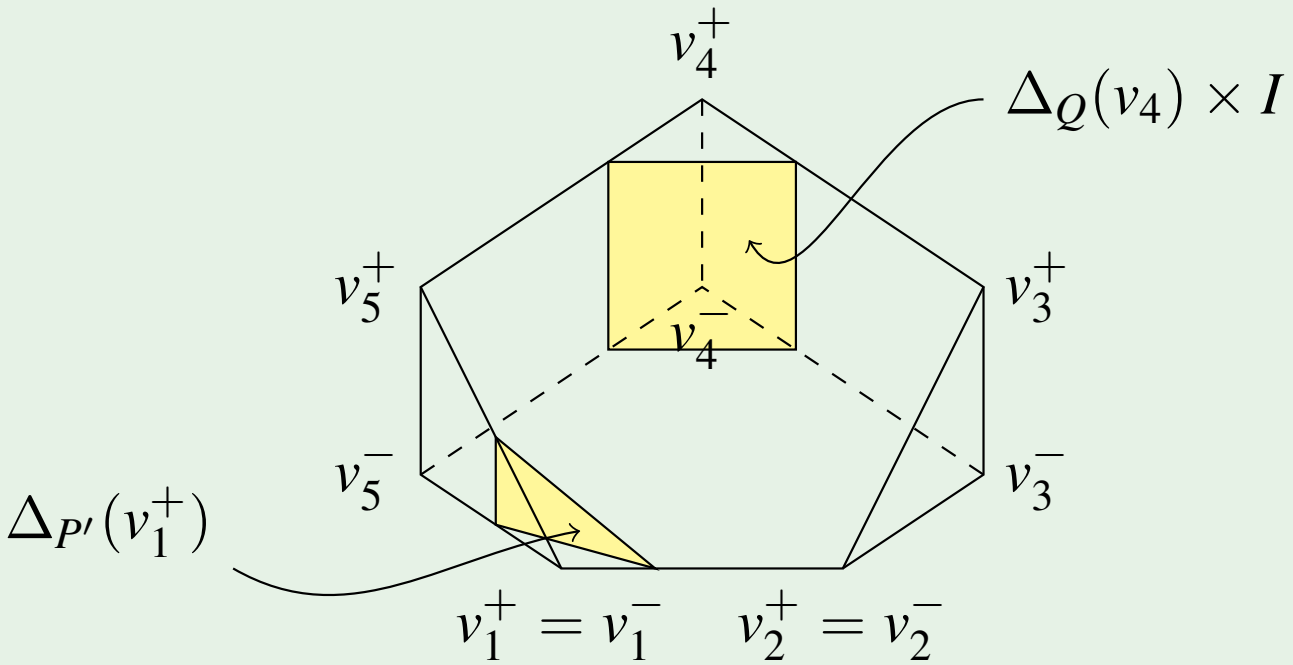
Example (Wedge of 5-gon)



Retraction-like decomposition

Lemma

- $v \in V(F_i)$,
- $\pi' : X(Q', \lambda') \to P'$ *orbit map.*

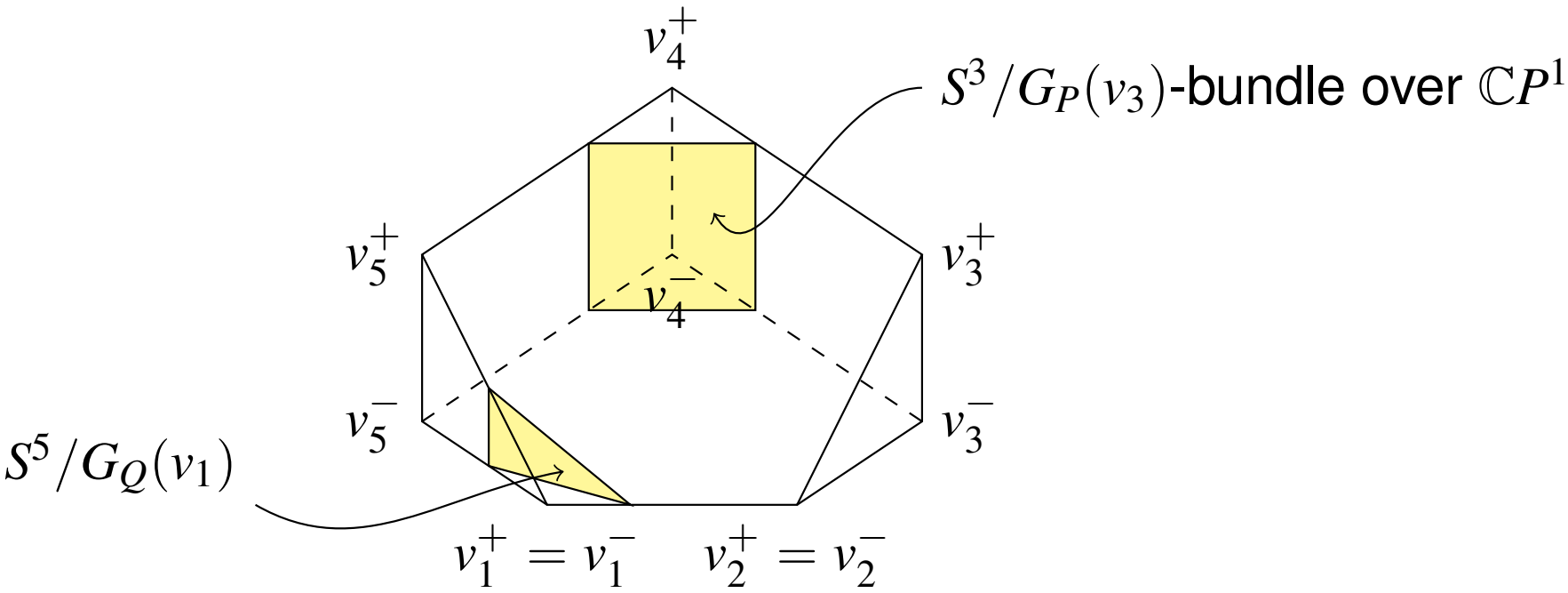
Then, $\pi'^{-1}(\Delta_{P'}(v^+)) \cong S^{2n+1}/G_{P'}(v^+)$. *Moreover,* $G_{P'}(v^+) \cong G_P(v)$.

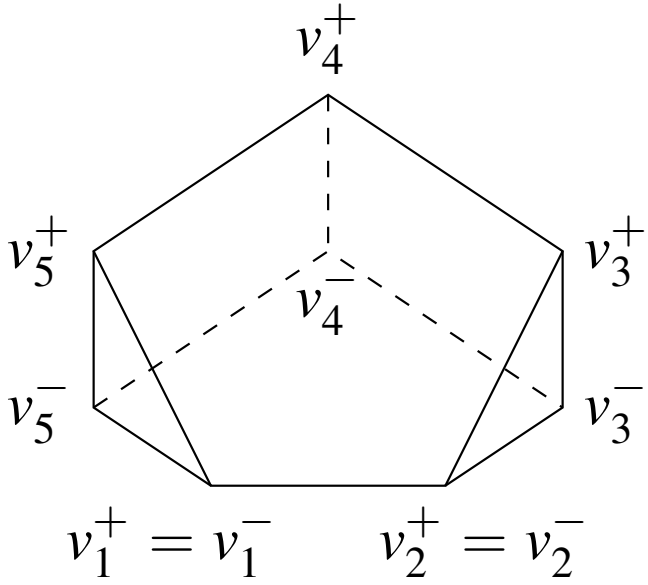
Lemma

- $v \in \backslash V(F_i)$,
- $\pi' : X(P', \lambda') \to P'$ *orbit map.*

Then, $\pi'^{-1}(\Delta_{P'}(v^{\pm}))$ *is a* $\pi^{-1}(\Delta_P(v))$*-fibration over* S^2.

Revisit: Wedge of 5-gon





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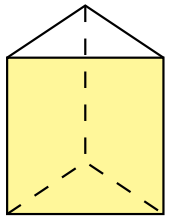
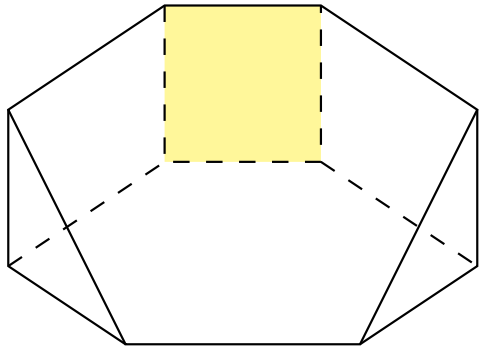
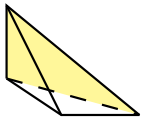
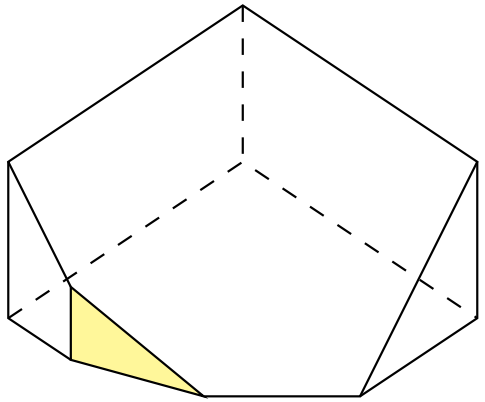


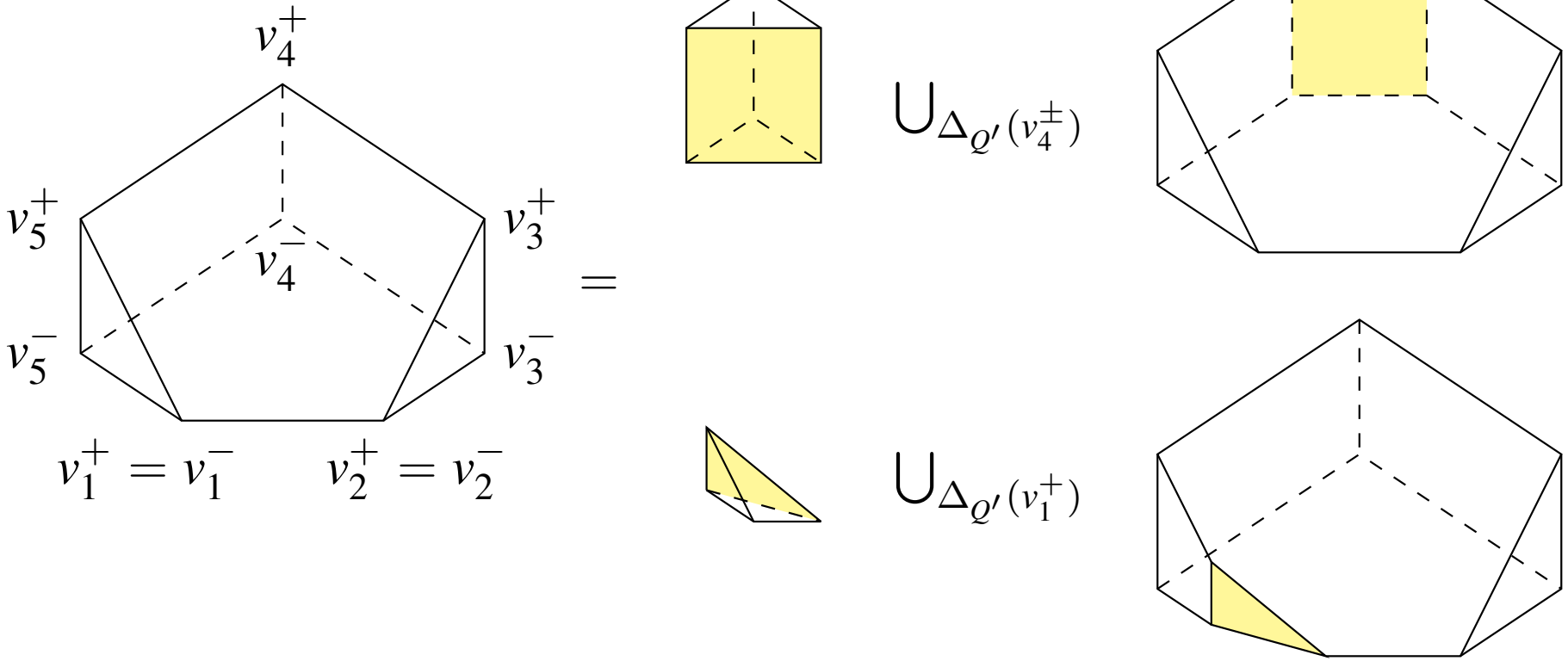
$\bigcup_{\Delta_{Q'}(v_4^{\pm})}$





$\bigcup_{\Delta_{Q'}(v_1^{+})}$





Theorem

If (P, λ) satisfies the "gcd assumption", then $H^*(X(P', \lambda'); \mathbb{Z})$ is torsion free and concentrated in even degrees.

Final remark

Please ask me, if you need a non-trivial example of orbifold X with $H^{odd}(X;\mathbb{Z})$ in arbitrary dimension.

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THANK YOU FOR YOUR ATTENTION!