Wedge operation and classification of toric spaces

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Toric spaces and Characteristic functions

- $K$: abstract $(n - 1)$-dim. simplicial (or precisely PL) sphere
- $V(K)$: set of vertices of $K$

**Definition (Characteristic functions)**

A **characteristic function** over $K$ is a map $\lambda: V(K) \rightarrow \mathbb{Z}^n$ such that for every maximal face $\{v_1, \ldots, v_n\}$ of $K$,

$$\det(\lambda(v_1) \cdots \lambda(v_n)) = \pm 1.$$ 

A **mod 2 characteristic function** over $K$ is a map $\lambda: V(K) \rightarrow \mathbb{Z}_2^n$ such that for every maximal face $\{v_1, \ldots, v_n\}$ of $K$,

$$\det(\lambda(v_1) \cdots \lambda(v_n)) = 1.$$ 

The pair $(K, \lambda)$ is called a **characteristic pair** and $K$ is the underlying complex.
Toric spaces and Characteristic functions

Let us consider the following families (up to D-J equivalence):

- Toric manifolds,
- Omnioriented quasitoric manifolds,
- Omnioriented topological toric manifolds with restricted $(S^1)^n$-action.

They bijectively correspond to:

- Characteristic functions which come from fans,
- Characteristic functions over polytopal spheres,
- Characteristic functions over star-shaped spheres, resp,

up to change of basis of $\mathbb{Z}^n$. Toric spaces are these kinds of manifolds determined by chr. ftns.
Classification of toric spaces

(1) Topological classification (up to diffeomorphism)
- Very hard
- Done for: toric surfaces, Bott manifolds of stage $\leq 4$, generalized Bott manifolds of stage $\leq 2$.

(2) Classification up to D-J equivalence
- Still hard but combinatorial
- Equivalent to classification of $(K, \lambda)$ (up to basis change)
- Done for: toric surfaces, generalized Bott manifolds of any stage, toric manifolds with Picard number $\leq 3$.
- **Wedge operation** is very useful for this classification
Wedge operation

- $K$: simplicial complex on $V(K)$
- $K_1 \star K_2 = \{ \sigma_1 \amalg \sigma_2 \mid \sigma_1 \in K_1, \sigma_2 \in K_2 \}$: simplicial join

**Definition**

The (simplicial) wedge of $K$ at $v \in V(K)$ is

$$\text{wed}_v K = (I \star \text{Lk}_K\{v\}) \cup (\partial I \star (K \setminus \{v\})),$$

where $I$ is a 1-simplex whose vertices are $v_1$ and $v_2$. 

![Diagram showing wedge operation](image)
Wedge operation

\[ m = |V(K)| \]

A consecutive application of \( \text{wed} \) is denoted by \( K(J) = K(j_1, \ldots, j_m) \).

\[ K = K(1, \ldots, 1) \]
\[ \text{wed}_1 K = K(2, 1, \ldots, 1) \]
\[ \text{wed}_2(\text{wed}_1 K) = \text{wed}_1(\text{wed}_2 K) = K(2, 2, 1, \ldots, 1) \]
\[ \text{wed}_{11}(\text{wed}_1 K) \cong \text{wed}_{12}(\text{wed}_1 K) \cong K(3, 1, \ldots, 1) \]
Wedge operation

If $K = \partial P^*$ is polytopal, $P$ is a simple $n$-polytope and $\exists$ a dual wedge

$$F \rightarrow \text{wed}_F P$$

s.t. $\partial(\text{wed}_F P)^* = \text{wed}_F^* K$.

Observe:

- $\text{wed}_F P$ has two faces isomorphic to $P$.
- If $\text{wed}_F P$ admits a QTM $M = M(\text{wed}_F P, \Lambda)$, called a wedge manifold, then $M$ has two chr. submfds, or the sides of $M$,

$$M_1 = M(P, \lambda_1) \text{ and } M_2 = M(P, \lambda_2).$$

This is also true for non-polytopal $K$. 
In general, let us assume:

- $K$: $(n - 1)$-dim PL sphere, $v \in V(K)$
- $M = M(\text{wed}_v K, \Lambda)$: wedge manifold
- $M_1, M_2$: sides of $M$.

**Theorem (Choi-P)**

The D-J type of $M$ is uniquely determined by those $M_1$ and $M_2$. Moreover,

1. $M$ is a TTM $\iff$ $M_1$ and $M_2$ are TTM.
2. $M$ is a QTM $\iff$ $M_1$ and $M_2$ are QTM.
3. $M$ is a toric manifold $\iff$ $M_1$ and $M_2$ are toric manifolds.

**Theorem (Choi-P)**

The same holds when $\Lambda$ is a mod 2 chr. ftn. Moreover,

1. $M$ is an RTTM $\iff$ $M_1$ and $M_2$ are RTTM.
2. $M$ is a small cover $\iff$ $M_1$ and $M_2$ are small covers.
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A nonempty family $\mathcal{F}$ of toric spaces is wedge-closed if

1. The set of underlying complexes is closed under $\text{wed}$,
2. $M \in \mathcal{F} \iff M_1, M_2 \in \mathcal{F}$.

Then TTM, QTM, TM, RTTM, and small covers are wedge-closed. (What about others?)
Definition

Fix a wedge-closed family $\mathcal{F}$. The diagram of $K$, $D(K, \mathcal{F})$, is the triple $(V, E, S)$ s.t.

1. $V = \text{the set of D-J classes over } K \text{ in } \mathcal{F}$,
2. $E = \text{the set of D-J classes over } \text{wed}_v K \text{ in } \mathcal{F}, \ v \in V(K)$,
3. $S = \text{the set of D-J classes over } \text{wed}_w(\text{wed}_v K) \text{ in } \mathcal{F}, \ v \neq w \in V(K)$.

Note: The wedged manifold $M(\text{wed}_v K, \Lambda) \in E$, whose two sides are $M(K, \lambda_1)$ and $M(K, \lambda_2)$, corresponds to the colored edge

$$\lambda_1 \overset{v}{\longrightarrow} \lambda_2.$$

Similarly, $S$ is a set of some subsquares

$$\begin{array}{c}
\lambda_1 \overset{v}{\longrightarrow} \lambda_2 \\
\downarrow w \quad \downarrow w \\
\lambda_3 \overset{v}{\longrightarrow} \lambda_4
\end{array}$$
Diagram of $K$

Definition
Fix a wedge-closed family $\mathcal{F}$. The diagram of $K$, $D(K, \mathcal{F})$, is the triple $(V, E, S)$ s.t.

1. $V =$ the set of D-J classes over $K$ in $\mathcal{F}$,
2. $E =$ the set of D-J classes over $\text{wed}_v K$ in $\mathcal{F}$, $v \in V(K)$,
3. $S =$ the set of D-J classes over $\text{wed}_w (\text{wed}_v K)$ in $\mathcal{F}$, $v \neq w \in V(K)$.

Note: To compute $D(K, \mathcal{F})$, we only have to deal with finitely many spheres. Moreover, $V$, $E$, and $S$ are finite sets in mod 2 cases.

Theorem (Choi-P, Classification-by-Wedge)

$\exists$ a combinatorial algorithm to find all toric spaces over $K(J)$ in $\mathcal{F}$ using $D(K, \mathcal{F})$.

Note: wedge-closedness is needed for Classification-by-Wedge.
**Wedge and classification**

**Definition**
A simplicial complex $K$ is a **seed** if $K$ cannot be written as a simplicial wedge.

The basic strategy is
1. Find **seeds** $K$ supporting toric spaces
2. Apply **Classification-by-Wedge** for each seed $K$

Recall some notable classification results:
- smooth toric surfaces
- generalized Bott manifolds (by Choi-Masuda-Suh)
- toric manifolds with Picard number 3 (by Batyrev)
Wedge and classification

Recall some notable classification results:

- smooth toric surfaces
- generalized Bott manifolds (by Choi-Masuda-Suh)
- toric manifolds with Picard number 3 (by Batyrev)

Their underlying complexes are

- the polygon $\partial P_m$
- the dual of a product of simplices:

$$\partial(\Delta^{n_1} \times \cdots \times \Delta^{n_k})^* = \partial \Delta^{n_1} \star \cdots \star \partial \Delta^{n_k}$$

- $K^{n-1}$ with $(n + 3)$ vertices, resp.
Wedge and classification

1. Smooth toric surfaces over $\partial P_m$
2. Gen. Bott mfds over $\partial(\Delta^{n_1} \times \cdots \times \Delta^{n_k})^*$
3. Toric mfds of Picard number 3

Note:

1. (2) and (3) are wedge-closed. Seeds for (2) and (3) supporting toric manifolds are:
   - For (2): $\partial(I \times \cdots \times I)^*$ = the duals of cubes
   - For (3): $\partial(I^3)^*$, $\partial P_5$.

2. Classification-by-Wedge works for (2) and (3), reproving the original works.

3. For (3), CbW does not need projectivity, unlike Batyrev’s original work.
Some polytopes and spheres

<table>
<thead>
<tr>
<th>$n \backslash m$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\Delta^1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$P_3 = \Delta^2$</td>
<td>$P_4 = I^2$</td>
<td>$P_5$</td>
<td>$P_6$</td>
<td>$P_7$</td>
<td>$P_8$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$\Delta^3$</td>
<td>$\Delta^2 \times \Delta^1 = \text{wed} P_4$</td>
<td>$I^3$</td>
<td>$\text{wed} P_5$</td>
<td>$\text{vc}(I^3)$</td>
<td>$\text{wed} P_6$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>4</td>
<td>$\Delta^4$</td>
<td>$\Delta^3 \times \Delta^1$</td>
<td>$\Delta^2 \times \Delta^2$</td>
<td>$C(4, 7)^*$</td>
<td>$I^4$</td>
<td>$B^*$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td>$\Delta^5$</td>
<td>$\Delta^4 \times \Delta^1$</td>
<td>$\Delta^3 \times \Delta^2$</td>
<td></td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>

- **Colored** ones are seeds.
- It looks very natural to study wedges of polygons, since polygons are very basic family of seeds.
Wedges of polygons and wedge-closedness

Classification-by-Wedge works well to the polygons, at least for toric manifolds and small covers. (Note: toric manifolds and small covers are wedge-closed families.)

The classification (nontrivially) implies

**Theorem (Choi-P)**

*Every toric manifold over $\partial P_m(J)$ is projective.*

It gives a partial affirmative answer to the wedge-closedness question on projective toric mfds.

- $M = M(\text{wed}_v K, \Lambda)$: a wedge manifold
- $M_1 = M(K, \lambda_1), M_2 = M(K, \lambda_2)$: two sides of $M$

**Question (Wedge-closedness question on PTM)**

$M$ is a PTM $\iff M_1$ and $M_2$ are PTM?
The classification also implies:

**Corollary (Failure of wedge-closedness of RTM)**

*For sufficiently large $m$, $\exists$ a small cover $M$ over $\text{wed}_v P_m$ which is not a RTM even though its two faces are RTM.*

**Sketch of proof.**

- All small covers over $P_m$ are RTM except
  \[
  \begin{pmatrix}
  1 & 0 & 1 & 0 & \cdots & 1 & 0 \\
  0 & 1 & 0 & 1 & \cdots & 0 & 1
  \end{pmatrix}.
  \]

- On the other hand, there are much fewer RTM over $\text{wed}_v P_m$ than small covers.
Seeds for each $m - n$

<table>
<thead>
<tr>
<th>$n \backslash m$</th>
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<th>3</th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\Delta^1$</td>
<td> </td>
<td> </td>
<td> </td>
<td> </td>
<td> </td>
<td> </td>
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<td>$P_8$</td>
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<td>$\Delta^3$</td>
<td>$\Delta^2 \times \Delta^1$ = wed $P_4$</td>
<td>$I^3$ wed $P_5$</td>
<td> </td>
<td>vc($I^3$) wed $P_6$</td>
<td> </td>
<td>\ldots</td>
</tr>
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<td>$\Delta^3 \times \Delta^1$</td>
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<td>$\Delta^4 \times \Delta^1$ &amp; wed $P_5$</td>
<td>$I^4$ &amp; $B^*$</td>
<td> </td>
<td>\ldots</td>
</tr>
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<td> </td>
<td>$\Delta^4 \times \Delta^1$ &amp; $\Delta^3 \times \Delta^2$</td>
<td> </td>
<td> </td>
<td> </td>
<td>\ldots</td>
</tr>
</tbody>
</table>

Note:

<table>
<thead>
<tr>
<th>$m - n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;good&quot; seeds</td>
<td>$\Delta^1$</td>
<td>$I^2$</td>
<td>$I^3, P_5, C(4, 7)^*$</td>
<td>???</td>
</tr>
</tbody>
</table>
**Finiteness of good seeds**

\( K: (n - 1) \)-dim. PL sphere with \( m \) vertices

**Definition (Good seeds)**

A seed \( K \) is **good** if \( K \) admits a mod 2 chr. ftn. or equivalently,
\[
s_{\mathbb{R}}(K) = m - n.
\]

**Theorem (Choi-P)**

For fixed \( m - n \geq 3 \), \( \exists \) only **finitely many good seeds** \( K^{n-1} \) with \( m \) vertices. More precisely,
\[
m \leq 2^{m-n} - 1.
\]

**Note:** there is a hierarchy of good seeds. (\( s(K) = m - n \), Polytopal, supporting toric mfds, etc.)
Finiteness of good seeds

\(K\): \((n - 1)\)-dim. PL sphere with \(m\) vertices

**Theorem (Choi-P)**

For fixed \(m - n \geq 3\), \(\exists\) only finitely many good seeds \(K^{n-1}\) with \(m\) vertices. More precisely,

\[ m \leq 2^{m-n} - 1. \]

**Note:** The theorem proves the following conjecture of Batyrev. The theorem implies that, for the smooth case, the analogue of Batyrev’s conjecture holds regardless of the geometry.

**Conjecture (Batyrev (1991))**

For any toric manifold \(X_\Sigma\) of Picard number \(\rho\), there exists a constant \(N(\rho)\) depending only on \(\rho\) such that the number of primitive collections in \(G(\Sigma)\) is less than \(N(\rho)\).
Future works

\[ K^{n-1} \text{ with } m \text{ vertices} \]

<table>
<thead>
<tr>
<th>( m - n )</th>
<th>1</th>
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<tbody>
<tr>
<td>good seeds</td>
<td>( \Delta^1 )</td>
<td>( I^2 )</td>
<td>( I^3, P_5, C(4, 7) )</td>
<td>???</td>
</tr>
<tr>
<td># of good seeds</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>&gt; 1000?</td>
</tr>
</tbody>
</table>

The classification of toric manifolds of Picard number 4 deals with the case \( m - n = 4 \). In spite of finiteness, we expect over thousands of good seeds. This is a serious ongoing project which heavily involves:

1. Enumeration of simplicial spheres: one should search for \((n - 1)\)-spheres with \((n + 4)\) vertices when \( n = 2, 3, \ldots, 11 \). This is almost hopeless without the good seed property.
2. Theory of oriented matroids: to determine polytopality of spheres. Also highly nontrivial.
3. Classification-by-Wedge: for further future. Worth to try first for specific interesting examples like \( \text{vc}(I^3) \).
Thank you for attention!
Have a nice time in Jeju.

감사합니다 谢谢
ありがとうございます

References
- Small covers over wedges of polygons, in preparation.