Geometrically bounding 3-Manifolds

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1. Abstract and Background
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abstract

It is well known that an arbitrary closed orientable 3-manifold can always be realized as a boundary of a 4-manifold. However, there are very rare closed hyperbolic 3-manifold M, which is the boundary of a totally geodesic hyperbolic 4-manifold. In this talk, Let v be the volume of a regular right-angled hyperbolic dodecahedron in $\mathbb{H}^3$, for each $n \in \mathbb{Z}_+$, we show there are at least $\lceil \frac{5n+3}{2} \rceil$ closed hyperbolic 3-manifolds of volume $16nv$ which bounding a totally geodesic hyperbolic 4-manifold. The proof uses small cover theory over a sequence of line-glued dodecahedra.

Survery

What kind of closed n-manifold can bound a (n+1)-manifold

- Rohlin (1951) $\Omega_3 = 0$
- (1983) Every almost flat n-manifold bounds a (n+1)-manifold (conjectured by Farrell and Zdravkovska)
- (1983) Every flat n-manifold M is a cusp section of a hyperbolic $(n+1)$-manifold.(conjectured by Farrell and Zdravkovska)
- Long-Reid(2000) gave a negative answer: If a hyperbolic closed 3-manifold M is geometrically bounding, then $\eta(M) \in \mathbb{Z}$.
- Meyerhoff-Neumann(1992):The set of η-invariants of all hyeprbolic 3-manifolds is dense in $\mathbb{R}$

Geometrically bounding manifold

Definition

If a hyperbolic n-manifold M is the totally geodesic boundary of a hyperbolic $n+1$ manifold N, then we say
M **bounds geometrically** or M is a **geometrically bounding** hyperbolic manifolds.

question

For a hyperbolic closed 3-manifold M, is there a hyperbolic totally geodesic 4-manifold N such that $\partial N = M$?

Jorgensen-Thurston's Dehn surgery theory(1978): given $x \in \mathbb{R}_+$, there are only finitely many hyperbolic 3-manifold with volume x. considering consider the function:

$$f(x) = \sup\{n | n \textit{ different } 3-\textit{manifold } M_n \textit{ with same volume } v \leq x\}$$

- Jorgensen-Thurston theory(1978): $f(x)$ is finite
- Millichap(2015):$f(x)$ grows at least factorially.

Main result

Theorem

Let v be the volume of a regular right-angled hyperbolic dodecahedron in $\mathbb{H}^3$, for each $n \in \mathbb{Z}_+$, there are at least $\lceil \frac{5n+3}{2} \rceil$ closed hyperbolic 3-manifolds of volume $16nv$, each of them bounding a totally geodesical hyperbolic 4-manifold.

Polyhedral product

Let K be an abstract simplicial complex with ground set $[m] := \{1, 2, \ldots, m\}$. $\emptyset \in K \subset 2^{[m]}$. $\sigma \in K \Rightarrow \tau \in K$, $\forall \tau \subset \sigma$
Associate to K m pairs of topological space :

$$(\underline{X}, \underline{A}) := (X_i, A_i)_{i=1}^m$$

. Then the corresponding polyhedral product is given by

$$(\underline{X}, \underline{A})^K := \bigcup_{\sigma \in K} D(\sigma),\ D(\sigma) = \prod_{i=1}^m Y_i = \begin{cases} X_i & \text{if} \quad i \in \sigma \\ A_i & \text{if} \quad otherwise \end{cases}$$

That is to say, for every $x = (x_i)_{i=1}^m \in \prod_{i=1}^m X_i$, define σ_x=$\{\text{i} \mid x_i \in X_i \backslash A_i\}$, then

$$(\underline{X}, \underline{A})^K = \{x \in \prod_{i=1}^m X_i | \sigma_x \in K\}$$

.

Real moment-amgle complex

- If (X_i, A_i) are the same pair (X, A), $(\underline{X}, \underline{A})^K$ is abbreviated as $(X, A)^K$.
- Specially, $(D^1, S^0)^K$ is defined as real moment-angle complex, denoted by $\mathbb{R}\mathcal{Z}_K$.

Example

Let $K = 2^{[3]} \backslash \{1, 2, 3\}$. The maximal simplices of K are

$$\{1, 2\}, \{2, 3\}, \{1, 3\}. \text{ Then we have}$$

$$\begin{aligned}\mathbb{R}\mathcal{Z}_K &= (D^1, S^0)^K \\ &= D^1 \times D^1 \times S^0 \bigcup S^0 \times D^1 \times D^1 \bigcup D^1 \times S^0 \times D^1 \\ &= \partial(D^1 \times D^1 \times D^1) \\ &\cong S^2\end{aligned}$$

Topological manifold realization

By Davis(2008) and L. Cai(2013)

$\mathbb{R}\mathcal{Z}_K$ is a topological n-manifold iff K is a GHS^{n-1} with $\pi_1(|K|) = 0$ when $n - 1 \geqslant 2$.

When assuming P^n to be a n-dim simple polytope and K^{n-1} is the dual of the boundary of P^n.$\mathbb{R}\mathcal{Z}_K$ is definitely to be a topological n-manifold.

Reconstruction procedure

- $\mathcal{F}(P^n) = \{F_1, F_2, \ldots, F_m\}$ Set of 1-codim faces of P^n.
- $\{e_1, e_2, \ldots, e_m\}$ is a basis in $\mathbb{Z}_2^m$.
- $\lambda : \mathcal{F}(P^n) = \{F_1, F_2, \ldots, F_m\} \longrightarrow \mathbb{Z}_2^m \ F_i \mapsto e_i$.

Obseravation

when $F_1, F_2, \ldots, F_n$ sharing a commom vertice $\lambda(F_1), \lambda(F_2), \ldots, \lambda(F_n)$ generate $\mathbb{Z}_2^n$.

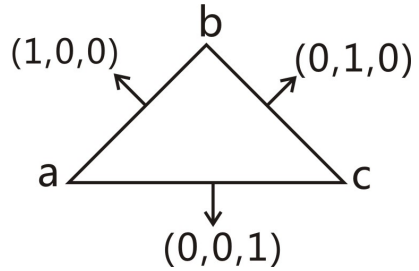
Constructing $M(P^n, \lambda) := P^n \times \mathbb{Z}_2^m / \sim$ by equivelant relation:

$$(p, g) \sim (q, h) \Longleftrightarrow \begin{cases} p = q \quad g = h & \text{if} \quad p \in \mathit{IntP} \\ p = q \quad g^{-1}h \in G_F & \text{if} \quad p \in \partial P \end{cases}$$

There is a unique face $F^k = F_{i_1} \cap \cdots \cap F_{i_{n-k}}$ that contain p as a interior point. G_F is the subgroup generated by $\lambda(F_{i_1}) \ldots \lambda(F_{i_{n-k}})$. It is easily to proof that $M(P^n, \lambda)$ is exactly $\mathbb{R}\mathcal{Z}_K$.

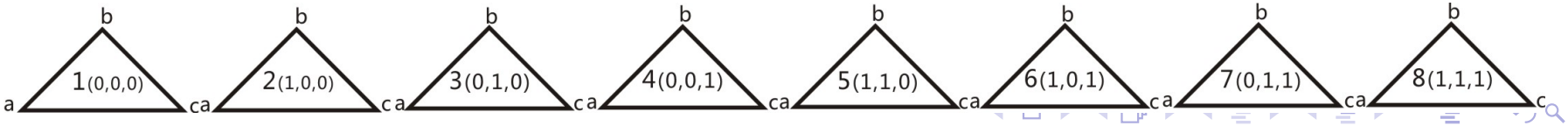
Example

Defining a characteristic function on simple polytope $P = \triangle^2$, $(\partial P)^*$ is exactly K, which was mentioned in Example 1



$$\lambda : \{\{a,b\},\{b,c\},\{a,c\}\} \to \{(1,0,0),(0,1,0),(0,0,1)\}$$
$$(a,b) \mapsto (1,0,0)$$
$$(b,c) \mapsto (0,1,0)$$
$$(a,c) \mapsto (0,0,1)$$

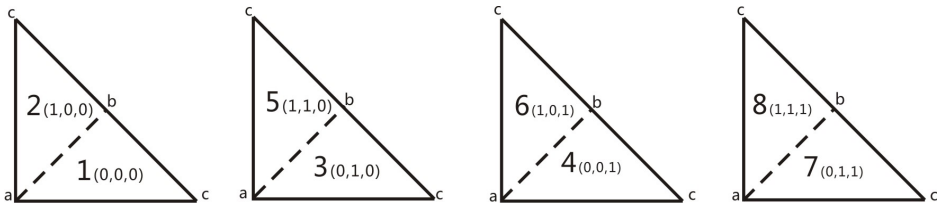
Then we will have 8 pieces, namely $\triangle^2 \times \mathbb{Z}_3^2$, shown as below:



Example

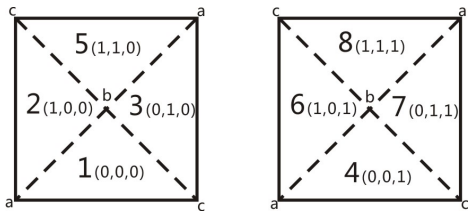
If $p \in (a,b)$, $(x_1,g_1) \sim (x_2,g_2) \Longleftrightarrow \begin{cases} x_1 = x_2 \ (\textit{same location}) \\ g_1 - g_2 \in \{(1,0,0)\} \end{cases}$

gluing 1 and 2, 3 and 5, 4 and 6, 7 and 8 along the edge(a,b):



If $p \in (b,c)$, $(x_1,g_1) \sim (x_2,g_2) \Longleftrightarrow \begin{cases} x_1 = x_2 \ (\textit{same location}) \\ g_1 - g_2 \in \{(0,1,0)\} \end{cases}$.

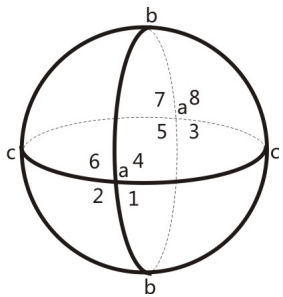
gluing 1 and 3, 2 and 5, 4 and 7, 6 and 8 along the edge(b,c):



Example

If $p \in (a, c)$, $(x_1, g_1) \sim (x_2, g_2) \Longleftrightarrow \begin{cases} x_1 = x_2 \ (\textit{same location}) \\ g_1 - g_2 \in \{(0, 0, 1)\} \end{cases}$.

gluing 1 and 4, 2 and 6, 3 and 7, 5 and 8 along the edge(a, c):



Thus $M(P, \lambda) \cong S^2 \cong \mathbb{R}\mathcal{Z}_K$

Buchstaber invariant and real toric manifolds

a natural $\mathbb{Z}_2^m$-action $\curvearrowright \mathbb{R}\mathcal{Z}_K$, where $m = |\mathcal{F}(P^n)|$

Buchstaber invariant

$S_R(P^n) = max\{rank\ H \mid H < Z_2^m \mid H(free) \curvearrowright R\mathcal{Z}_K\}$

$$\begin{array}{ll} \mathbb{Z}_2{}^m/H \curvearrowright & \mathbb{R}\mathcal{Z}_K/H \\ & \downarrow \\ & P^n \end{array} \quad \text{real toric manifolds}$$

If $S_R(P^n) = m - n$, $\mathbb{R}\mathcal{Z}_K/H^{(m-n)}$ is named as small cover.

Characteristic function and linear-independent condition

Assuming the rank of a freely acting subgroup H is r.

$$1 \to H \hookrightarrow \mathbb{Z}_2^m \to \mathbb{Z}_2^m/H \to 1 \quad \textit{exact}$$
$$0 \to \mathbb{Z}_2^r \hookrightarrow \mathcal{F}(P^n) \xrightarrow{\lambda_H} \mathbb{Z}_2^{m-r} \to 0 \quad \textit{exact}$$

Remark

By **Four Color Theorem**, *if* $m-n \leqslant 3$ *or* $n \leqslant 3$, $S_R(P^n) = m-n$.

Remark

free acting requirment $\xrightarrow{\textit{ensuring}}$ *linear-independent condition always stands at every vertices.*
i.e. for each vertex $v = F_{i1} \cap F_{i2} \cap \cdots \cap F_{in}$, $\lambda_H(F_{i1}), \ldots, \lambda_H(F_{in})$ *can extend a basis of* $(\mathbb{Z}_2)^n$.

1-1 correspongding

$$(P^n, \lambda) \to \mathbb{R}\mathcal{Z}_K \xrightarrow{\textit{be parallelly applied to}} \text{all real toric manifolds.}$$

1-1 correspongding

$$\{\mathbb{R}\mathcal{Z}_P/H, H \in \mathcal{F}(Z_2)^m\} \Longleftrightarrow \{(P^n, \lambda_H)\}$$

We always use $M(P^n, \lambda_H)$ to denote the corresponding real toric manifold.

Topology of $\mathbb{R}\mathcal{Z}_K/H$

M. W. Davis and T. Januszkiewicz. (1991)

$\mathbb{Z}_2$-coefficient cohomology of small cove depends only on the underlying polytope and its characteristic function.

L. Cai. (2013)

$\mathbb{Z}$-coefficient cohomology of $\mathbb{R}\mathcal{Z}_K$.

Alexander. I. Suciu. The rational (2013)

rational homology of real toric manifolds

Suyoung Choi and Hanchul Park. (2013)

a formular of the cohomology of real toric manifold
(a combinatorial version of Hoster Theorem)

1. K: simplicail complex on $[1, 2, \ldots, m]$.
2. $\varphi : \mathbb{Z}_2^m \longrightarrow 2^{[m]}$ a bijective map s.t the i-th entry of $v \in \mathbb{Z}_2^m$ is nonzero iff $i \in \varphi(v)$.
3. λ: a $\mathbb{Z}_2^n$-coloring characteristic function.
4. $\Lambda_{(n\times m)} = (\lambda(F_1) \ldots \lambda(F_m))$: characteristic matrix
5. $row\Lambda$: $\mathbb{Z}_2$-space generated by the n rows of Λ.

Choi-Park (2013)

$$H^p(M(K,\lambda); G) \cong \bigoplus_{\varphi^{-1}(\omega)\in row\Lambda, \omega\subseteq[m]} \widetilde{H}^{p-1}(K_\omega; G)$$

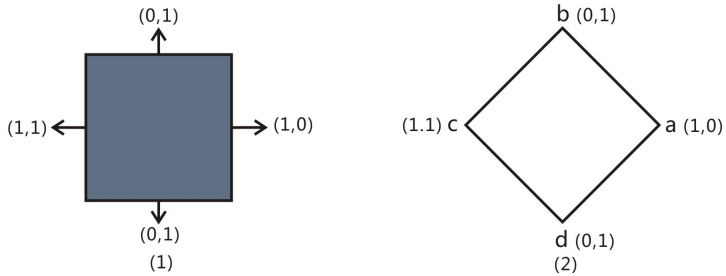
where K_ω is sub-complex of K by restricting to $\omega \subset [m]$. Moreover

$$\beta^i(M(K,\lambda); G) = \sum_{\omega\in row\lambda} \widetilde{\beta}^{i-1}(K_\omega; G)$$

when $\omega = 0$, $K_\omega = \emptyset$, $\beta^{-1}(K_\omega) = 1$.

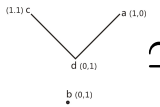
Calculate the Betti number of Klein Bottle $S = M(P, \lambda)$.

A colored P is (1) and corredponding colored $K = (\partial P)^*$ is (2)

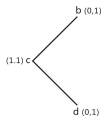


$Row\Lambda$=<(1,0,1,0),(0,1,1,1)>=((0,0,0,0,),(1,0,1,1),(0,1,0,1),(1,1,1,0))

$\omega_1 = (0,0,0,0)$, $K_{\omega_1} = \varnothing$. By definition $\beta^{-1}(K_{\omega_1}) = 1$

$\omega_2 = (1,0,1,1)$, $K_{\omega_2}=$  $\simeq \cdot$ so $\widetilde{\beta}^0(K_{\omega_2}) = 0$

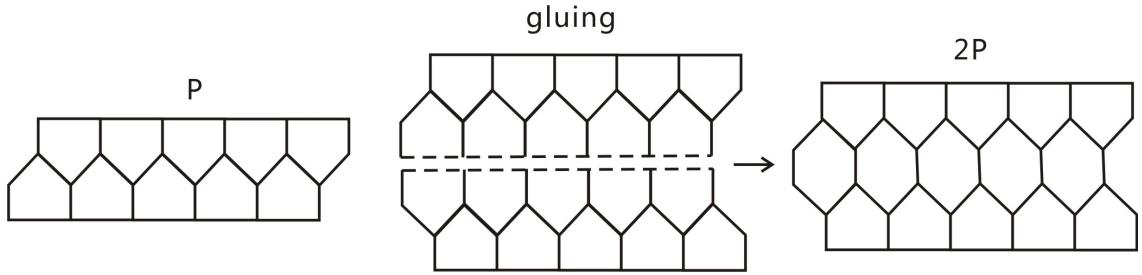
$\omega_3 = (0,1,0,1)$, $K_{\omega_2} =$ $\simeq S^0$ so $\widetilde{\beta}^0(K_{\omega_3}) = 1$

$\omega_4 = (1,1,1,0)$, $K_{\omega_2}=$  $\simeq \cdot$ so $\widetilde{\beta}^0(K_{\omega_4}) = 0$

Then β^0(S)=1, β^1(S)=1, β^2(S)=0

Object polytopes nP

P :regular right-angled hyperbolic dodecahedron in $\mathbb{H}^3$.
$nP(n \in \mathbb{Z}_+)$:right-angled hyperbolic polytopes constructed by linearly-glued of n many P, $|\mathcal{F}(nP)| = 5n + 7$

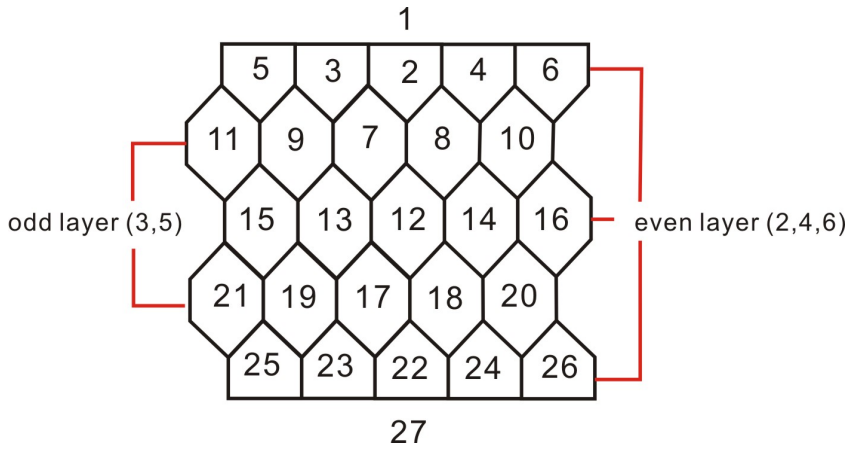


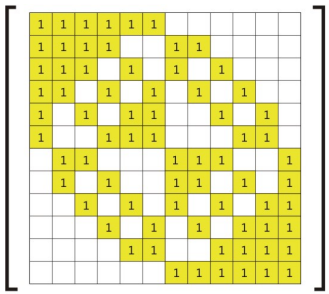
Definition

For a simple polytope Q, K is the dual of ∂Q. Denoting by $M_K = M(a_{ij})_{m\times m}$ the adjacent matrix of simplicial complex K, $(m = |\mathcal{F}(Q)| = |\{vertices\ of\ K\}|)$. That means

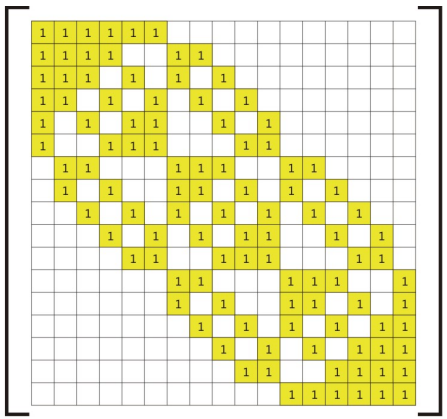
$$a_{ij} = \begin{cases} 1 & \text{vertex i and j are connected by 1-simplex in K} \\ 0 & \text{otherwise} \end{cases}$$

Object polytopes *nP*

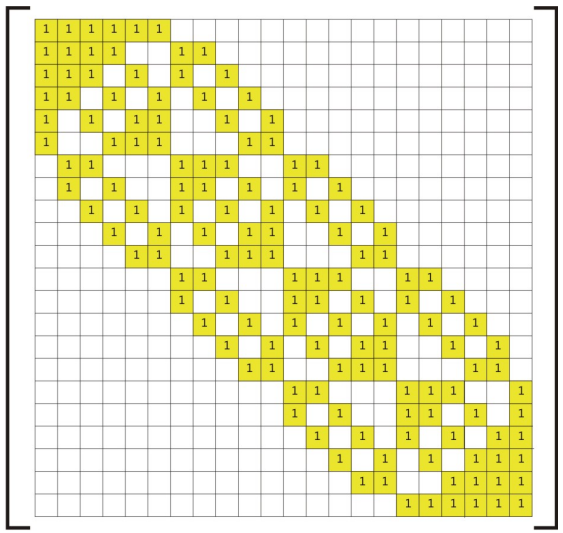




(1)



(2)



(3)

Equivatient homeomorphism

Definition

For two G-manifold M_1 and M_2, an equivarient homeomophism is an homeomorphism $f : M_1 \to M_2$ together with for all $g \in G$ and $x \in M_1$, $f(gx) = gf(x)$. Those two G-manifolds are said to be equivariently homeomorphic if such map exists.

$\mathbb{A} = (A(P))$ is symmetric group of P^n and $G = GL_n(\mathbb{Z})$. Small covers $M(P^n, \lambda)$ and $M(P^n, \mu)$ are equivariently homeomorphic if $\lambda = g \circ \mu \circ a$ for some $a \in \mathbb{A}$ and $g \in GL_n(\mathbb{Z})$

- Fix the color of the first three facets to be e_1, e_2, e_3

1. $1P$:2165
2. $2P$:392170
3. $3P$:57507285

Notation remark

encoding every $\mathbb{Z}_2^*$-coloring vector by an integer through binary.A characteristic matrix can also be viewed as a characteristic vector.

For example 2

- the characteristic matrix:

$$(\lambda(\mathcal{F}_1), \lambda(\mathcal{F}_2), \lambda(\mathcal{F}_3)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- corresponding characteristic vector C: (1,2,4),
- rowΛ=$\langle(1,0,0),(0,1,0),(0,0,1)\rangle$.

Admissible extending

Definition

Starting from a $\mathbb{Z}_2^3$-coloring λ on nP, we can have $2^m - 1$ $(\mathbb{Z}_2)^4$-colorings on nP by adding a non-zero fouth row to the $3 \times m$ characteristic matrix Λ of λ as shown below, where $m = 5n + 7$ and $* \in \{0, 1\}$. Those characteristic functions are called the extendings of λ and they naturally satisfy the "linear-independent" condition.

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 1 \\ 0 & 1 & 0 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ * & * & * & \dots & * \end{pmatrix}$$

Admissible extending

Definition

A $\mathbb{Z}_2^3$-coloring λ on nP is addmissible if there is an extending $\mathbb{Z}_2^4$-coloring δ satisfying $M(nP, \lambda)$ is unorientable while $M(nP, \delta)$ is orientable. Such δ is called an admissible extending of λ.

Orientablility of real toric manifold

Theorem H.Nakayama-Y.Nishimura (orientablility of small cover)

For a basis $\{e_1 ... e_n\}$ of $\mathbb{Z}_2^n$, $\epsilon : (\mathbb{Z}_2)^n \to \mathbb{Z}_2 = \{0, 1\}$ is a homomorphism defined by $\epsilon(e_i) = 1$(i=1,...,n). A small cover $M(P^n, \lambda)$ is orientable if and only if there exists a basis $\{e_1, ..., e_n\}$ of $\mathbb{Z}_2^n$ such that the image of $\epsilon\lambda$ is $\{1\}$.

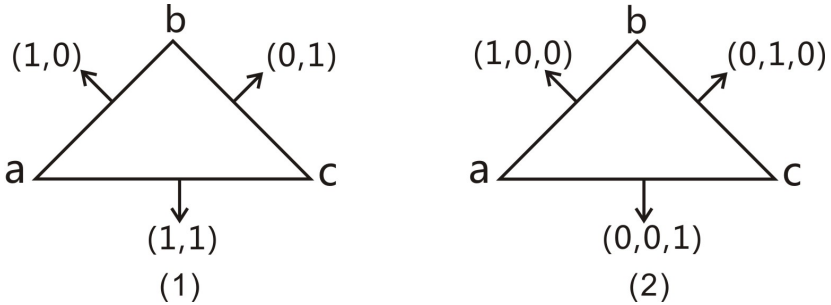
such theorem can be adjusted to meet all real toric manifolds instead of only small covers by merely parallell generelization.

Orientablility of real toric manifold

Remark

Every admissible $(\mathbb{Z}_2)^3$-coloring λ over P^n has only an uniquely admissible extending $(\mathbb{Z}_2)^4$-coloring δ whose characteristic matrix satisfying that the sum of every column is $1 \mod 2$.

Example(Uniqeness of admissible extending)



λ and δ are the characteristice fuction of (1) and (2) respectively. Then $M(\Lambda, \lambda) \cong RP^2$ is unorientable and $M(\Delta, \lambda) \cong S^2$ is orientable. Here δ is the unique admissible extending of δ whose characteristic matrix satisfies that every colum is 1 mod 2

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}_{\lambda} \xrightarrow{extending} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}_{\delta}$$

- $1P$:2165 (2155) / $2P$:392170 (392130) /
- $3P$:57507285 (57507085)

main result

Theorem

Let v be the volume of a regular right-angled hyperbolic dodecahedron in $\mathbb{H}^3$, for each $n \in \mathbb{Z}_+$, there are at least $\lceil \frac{5n+3}{2} \rceil$ closed hyperbolic 3-manifolds of volume $16nv$, each of them bounding a totally geodesical hyperbolic 4-manifold.

Proof of main result

For an admissible $\mathbb{Z}_2^3$-coloring λ on nP, there is a natural $\mathbb{Z}_2^4$-coloring δ on nP. Thus we have two 3-manifolds $M(nP, \delta)$ and $M(nP, \lambda)$, where $M(nP, \delta)$ is the orientable double cover of $M(nP, \lambda)$.

- Firstly, two steps to show $M(nP, \delta)$ is geometrically bounding:

1. extends the coloring δ to a $\mathbb{Z}_2^5$-coloring ε on the 120-cell such that $N(nL, \varepsilon)$ is an orientable 4-manifold in which $M(nP, \lambda)$ can be embedded in.
2. since $M(nP, \delta)$ is the orientable double cover of $M(nP, \lambda)$, it can be inferred that $M(nP, \delta)$ admite a fixed -point free orientation-reversing involuntion. Therfore $N - M(nP, \lambda)$ is a totally geodesic hyperbolic 4-manifold with boundary

- λ:$\mathbb{Z}_2^3$-coloring on nP
- δ:$\mathbb{Z}_2^4$-coloring on nP
- ε:$\mathbb{Z}_2^5$-coloring on nL (L:120-cell)

$$N(nL,\varepsilon)$$
embedded
$$\uparrow$$

$M(nP,\delta) \xrightarrow{d.c} M(nP,\lambda)$ $\quad\quad$ $\partial\,(N(nL,\varepsilon)\text{-}M(nP,\lambda))=M(nP,\delta)$

- secondly, calculate the lower bound

Remark 1

Let $n \in \mathbb{Z}_+$ be an even number, there is an admissible $\mathbb{Z}_2^3$-coloring λ, such that for the natural associated $\mathbb{Z}_2^4$-coloring δ, $\beta_1(M(nP, \delta)) = n + 1$.

Remark 2

For any $n \in \mathbb{Z}_+$ is an even number, there is an admissible $\mathbb{Z}_2^3$-coloring λ, such that for the natural associated $\mathbb{Z}_2^4$-coloring δ, $\beta_1(M(nP, \delta)) = 5n - 3$.

Remark 3

For any $n \in \mathbb{Z}_+$ is an even number, for any odd integer $k \in [5n - 1, 5n + 3]$, there is an $\mathbb{Z}_2^4$-coloring admissible extending δ over nP such that $\beta_1(M(nP, \delta)) = k$.

Remark 4

For any even integer $n \in \mathbb{Z}_+$ and odd integer $k \in [1, n-1]$, there is an $\mathbb{Z}_2^4$-coloring admissible extending δ over nP such that $\beta_1(M(nP, \delta)) = k$.

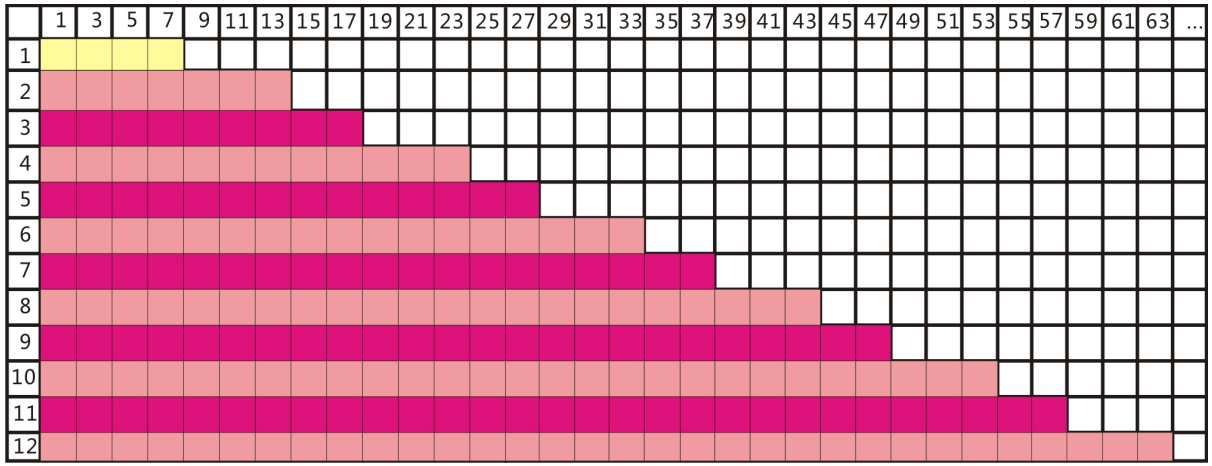
Remark 5

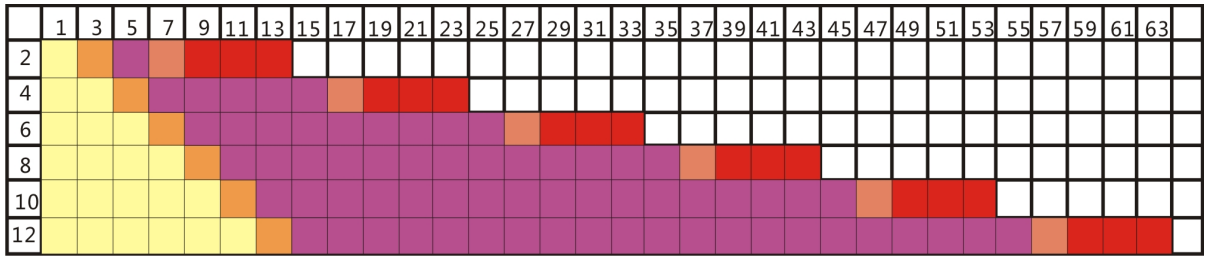
For any even integer $n \in \mathbb{Z}_+$ and any odd integer $k \in [n+3, 5n-5]$, there is an $\mathbb{Z}_2^4$-coloring admissible extending δ over nP such that $\beta_1(M(nP, \delta)) = k$.

Theorem

Let v be the volume of a regular right-angled hyperbolic dodecahedron in $\mathbb{H}^3$, for each even $n \in \mathbb{Z}_+$, there are at least $\lceil \frac{5n+3}{2} \rceil$ closed hyperbolic 3-manifolds of volume $16nv$, each of them bounding a totally geodesical hyperbolic 4-manifold.

- similarily, we can fill up the odd case and finish the whole theorem.





thank you for your attention