The cohomology ring of the symmetric square of quaternionic projective space

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References


Symmetric squares $SP^2(\mathbb{HP}^n)$ of quaternionic projective space

The symmetric square of $\mathbb{HP}^n$ is the quotient space

$$SP^2(\mathbb{HP}^n) = \mathbb{HP}^n \times \mathbb{HP}^n / \sim$$

by identifying $(z, v) \sim (v, z)$ where $z, v \in \mathbb{HP}^n$, that is

$$SP^2(\mathbb{HP}^n) = \mathbb{HP}^n \times \mathbb{HP}^n / \mathbb{Z}/2.$$
The symmetric square $SP^2(\mathbb{H}\mathbb{P}^n)$ can be decomposed as

$$SP^2(\mathbb{H}\mathbb{P}^n) = L_n \cup A_n \ N_n$$

where $A_n = L_n \cap N_n$ and $N_n$ is a closed neighbourhood of the diagonal

$$\triangle := \{ (z, z) \mid z \in \mathbb{H}\mathbb{P}^n \} \subset SP^2(\mathbb{H}\mathbb{P}^n),$$

also $L_n$ is the closure of the complement $SP^2(\mathbb{H}\mathbb{P}^n) \setminus N_n$. 
More about $A_n$, $L_n$ and $N_n$

- $N_n \simeq \triangle \simeq \mathbb{H}P^n$
- $L_n \simeq$ space of all unordered pairs of distinct points on $\mathbb{H}P^n$
- $A_n$ may be described as the total space of the (real)projectivisation $\mathbb{R}P(\tau_{\mathbb{H}P^n})$ of the tangent bundle of $\mathbb{H}P^n$,

$$\mathbb{R}P^{4n-1} \to A_n \to \mathbb{H}P^n.$$
Main results

\[ H^*(SP^2; \mathbb{Z}) := H^*(SP^2(\mathbb{H}P^\infty); \mathbb{Z}) \]

Theorem

The integral cohomology ring of \( SP^2(\mathbb{H}P^\infty) \) can be given by the following. Let \( s, \ell, i \geq 1 \), \( m \geq 0 \), \( 0 < j < 2i \), then

\[ H^*(SP^2; \mathbb{Z}) \cong \mathbb{Z} \left[ (1/2)^{s-1} h^s, \ (1/2)^m g^\ell h^m, \ t_{i,j} \right] / I \]

where

\[ I = \left( 2t_{i,j}, \ t_{i,j} \cdot t_{k,l}, \ t_{i,j} \cdot (1/2)^{s-1} h^s, \ t_{i,j} \cdot (1/2)^m g^\ell h^m \right) \]

and

\[ |g| = 4, \ |h| = 8, \ |t_{i,j}| = 4i + 2j + 1. \]
Main results (continue)

\[ H^*(SP^2_n; \mathbb{Z}) := H^*(SP^2(\mathbb{H}\mathbb{P}^n); \mathbb{Z}) \]

**Proposition**

The homomorphism \( \eta^* \), induced by inclusion \( \mathbb{H}\mathbb{P}^n \subset \mathbb{H}\mathbb{P}^\infty \)

\[ \eta^* : H^*(SP^2; \mathbb{Z}) \rightarrow H^*(SP^2_n; \mathbb{Z}) \]

is surjective; that is,

the integral cohomology ring of \( SP^2(\mathbb{H}\mathbb{P}^n) \) can be described as

\[ H^*(SP^2_n; \mathbb{Z}) \cong H^*(SP^2; \mathbb{Z}) / \ker \eta^* \]

We will explain \( \ker \eta^* \) later.
Main methods

From the decomposition of the symmetric square

\[ SP^2(\mathbb{HP}^n) = L_n \cup_{A_n} N_n, \]

and the fact \( SP^2_n / N_n \cong L_n / A_n \) we obtain:

1. the commutative ladder of cofibre sequences

\[
\begin{array}{cccccc}
N_n & \rightarrow & SP^2_n & \overset{u}{\rightarrow} & SP^2_n / N_n & \rightarrow \cdots \\
\uparrow & & \uparrow f & & \uparrow r \cong & \\
A_n & \rightarrow & L_n & \rightarrow & L_n / A_n & \rightarrow \cdots
\end{array}
\]

2. take \( n \rightarrow \infty \)

3. apply \( H^*(-) \) to get a map of long exact sequences (LES).

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$H^*(A) := H^*(A_{n \rightarrow \infty})$ and $H^*(L) := H^*(L_{n \rightarrow \infty})$

How to compute $H^*(A)$ and $H^*(L)$?

The following are useful,

- $A \simeq \mathbb{RP}^\infty \times \mathbb{HP}^\infty$,
- $L \simeq S^\infty \times \mathbb{Z}/2 (\mathbb{HP}^\infty \times \mathbb{HP}^\infty) =: B$,

then

$$H^*(B; \mathbb{Z}) \cong \mathbb{Z}[x, y, w]/(2w, xw), \quad |w| = 2, \ |x| = 4, \ |y| = 8$$

may be deduced from the fibration

$$\mathbb{RP}^4 \rightarrow B \rightarrow Gr_H(\infty, 2).$$

Note: Alternatively we can use Fred Roush’s (unpublished) results for $H^*(B; \mathbb{Z})$. 

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Definitions of $x, y$ and $w$

Recall
\[ H^* (\text{Gr}_H(\infty, 2); \mathbb{Z}) \cong \mathbb{Z}[p_1, p_2], \text{ where } |p_1| = 4, \, |p_2| = 8, \]
\[ H^*(\mathbb{RP}^\infty; \mathbb{Z}) \cong \mathbb{Z}[a]/(2a), \text{ where } |a| = 2. \]

**Definition**

Given the maps
\[ i : B \to \text{Gr}_H(\infty, 2), \quad \pi : B \to \mathbb{RP}^\infty, \]
generators $w, x, y \in H^*(B; \mathbb{Z})$ are defined by
\[ w := \pi^*(a), \quad x := i^*(p_1) + w^2, \quad y := i^*(p_2) + w^4. \]
Definitions of $g, h, \cdots$

**Definition**

Given $f : L \rightarrow SP^2$ from the ladder, and $B := L$,

$$f^* : H^*(SP^2; \mathbb{Z}) \rightarrow H^*(B; \mathbb{Z});$$

then the generators $(1/2)^{s-1}h^s, (1/2)^m g^\ell h^m \in H^*(SP^2; \mathbb{Z})$ are defined uniquely by

$$f^* : (1/2)^{s-1}h^s \mapsto 2y^s$$

$$f^* : (1/2)^m g^\ell h^m \mapsto x^\ell y^m$$

where $\ell \geq 1, \ m \geq 0$. 
Definition of $t_{i,j} \in H^*(SP^2; \mathbb{Z})$

Let $* = 4i + 2j + 1$, $i \geq 1$, and $0 < j < 2i$, then the generator $t_{i,j} \in H^*(SP^2; \mathbb{Z})$ is defined to be

$$t_{i,j} := u^* \circ (r^*)^{-1} \delta(z^i w^j),$$

where $\delta(z^i w^j) \in H^*(L/A; \mathbb{Z})$, $z \in H^4(HP^\infty; \mathbb{Z})$, $w \in H^2(\mathbb{RP}^\infty; \mathbb{Z})$.

For $* = 4k + 1$, $4k + 3$, the symmetric squares of $HP^n$ August 2014 12 / 21
The integral cohomology of $SP_2^n := SP^2(\mathbb{HP}^n)$

Lemma

The homomorphism $\eta^* : H^*(SP^2; \mathbb{Z}) \to H^*(SP_2^n; \mathbb{Z})$ is surjective.

Roughly speaking, $\ker \eta^*$ is explained by the fact

$$z^{n+1} = 0 \quad \text{in} \quad H^*(\mathbb{HP}^n; \mathbb{Z}) \cong \mathbb{Z}[z]/(z^{n+1});$$

for instance,

$$\eta^*(t_{i,j}) = 0, \quad \text{if} \quad i > n.$$
The integral cohomology of $SP_3^2$ can be expressed as

$$H^*(SP_3^2; \mathbb{Z}) \cong H^*(SP^2; \mathbb{Z})/\mathcal{I}$$

where

$$\mathcal{I} = \left(t_{i,j}, \ g^4 - 4 \cdot \frac{1}{2} g^2 h + \frac{1}{2} h^2, \ \frac{1}{2} g^3 h - 3 \cdot \frac{1}{4} gh^2, \ g^7, \ \frac{1}{8} h^4 \right)$$

for $i > 3$. 
\[ H^* := H^*(SP_3^2; \mathbb{Z}) \text{ with generators.} \]

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\( H^* = 0 \) for other \( \* \).
Table: $H^*(SP^2_3;\mathbb{Z})$

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$H^* = 0$ for other *.
The following product structures on $H^*(SP^2; \mathbb{Z})$ become visible after analysing the LESs.

Some of the product structures

- $t_{i,j}t_{k,l} = 0$,
- $(1/2)^{s-1}h^s \cdot (1/2)^{u-1}h^u = 2 \cdot (1/2)^{s+u-1}h^{s+u}$,
- $(1/2)^m g^\ell h^m \cdot (1/2)^{s-1}h^s = 2 \cdot (1/2)^{m+s} g^\ell h^{m+s}$,
- $(1/2)^m g^\ell h^m \cdot (1/2)^{i}g^i h^i = (1/2)^{m+j} g^\ell+i h^{m+j}$.

However in order to obtain the remaining multiplicative structure some other approach is needed!
Product structures: part II

How to find the products

\[ t_{i,j} \cdot (1/2)^m g^l h^m = ? \quad \text{and} \quad t_{i,j} \cdot (1/2)^{s-1} h^s = ? \]

Recall \( 2t_{i,j} = 0 \).

**Lemma**

The mod 2 reduction

\[ \rho : H^*(SP^2; \mathbb{Z}) \to H^*(SP^2; \mathbb{Z}/2) \]

is injective in dimensions \( * \neq 0 \mod 4 \).

Then repeat our method of LESs using additional input.
Cohomology with $\mathbb{Z}/2$ coefficients

And we get...

$$t_{i,j} \cdot \frac{1}{2}^{s-1} h^s = 0 \quad \text{and} \quad t_{i,j} \cdot \frac{1}{2}^m g^\ell h^m = 0$$

It is generally believed that computation of $\mathbb{Z}/2$-cohomology is relatively easier than integral cohomology (what I was told)!!

I have carried out the calculations, and they agree with Nakaoka’s work; they are also consistent with applying the Universal Coefficient Theorem to $H^*(SP^2_n; \mathbb{Z})$. 
There are several vital requirements for calculating the $\mathbb{Z}/2$-cohomology. These include:

- the Thom space $SP^2/N$
- the Steenrod squares
- $BPin(4)$ and more...

We will talk about these
There are several vital requirements for calculating the $\mathbb{Z}/2$-cohomology. These include:

- the Thom space $SP^2/N$
- the Steenrod squares
- $BPin(4)$ and more...

We will talk about these on 19 August 2014 at ICM in Seoul.
Thank you for listening.