The theory of $(2n, k)$-manifolds

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joint work with Victor M. Buchstaber, Steklov Mathematical Institute, Russian Academy of Sciences

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Content of the talk

- Motivating results
- Theory of toric \((2n, k)\) - manifolds
- Complex Grassmann and flag manifolds as \((2n, k)\) - manifolds
- Seminal examples of \((2n, k)\)-manifolds
The main ideas of our approach are published in:

Victor M. Buchstaber (joint with Svjetlana Terzić) 
\((2n, k)\)-manifolds and applications, Mathematisches Forschung Institut Oberwolfach, Report No. 27/2014, p. 5–8, DOI: 10.4171/OWR/2014/27
We consider the complex Grassmann manifold \( G_{4,2} = U(4)/U(2) \times U(2) \) and the canonical action \( T^4 \hookrightarrow G_{4,2} \) which induces effective action of \( T^3 \).

**Theorem**

\( X = G_{4,2}/T^3 \) is homeomorphic to the quotient space

\[
(\Delta_{4,2} \times \mathbb{CP}^1)/ \approx
\]

where \((x, c) \approx (y, c') \iff x = y \in \partial \Delta_{4,2} \).
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**Theorem**

$X = G_{4,2}/T^3$ is homeomorphic to the quotient space

$$(\Delta_{4,2} \times \mathbb{C}P^1)/\sim$$

where $(x, c) \sim (y, c') \iff x = y \in \partial \Delta_{4,2}$.

**Corollary**

$G_{4,2}/T^3$ is homeomorphic to the join $S^2 \ast S^2$ which is homotopy equivalent to $S^5$.

**Theorem**

$G_{4,2}/T^3$ is a topological manifold without boundary, and, thus, $G_{4,2}/T^3$ is homeomorphic to the sphere $S^5$. 
• $S^5$ has unique differentiable structure, the standard one;

• suggests: no differentiable structure on $X = G_{4,2}/T^3$ such that
  $\pi : G_{4,2} \to X$ is a smooth map;
  otherwise $X$ would be diffeomorphic to the standard sphere $S^5$,
  $S^1 \hookrightarrow S^5$ smoothly, while it is not clear where such an action on $X$
  would come from.

— We prove the quotient structure is not differentiable;

— Describe the corresponding smooth and singular points;
Consider representation $T^4 \rightarrow T^6$ given by

$$(t_1, t_2, t_3, t_4) \rightarrow (t_1 t_2, t_1 t_3, t_1 t_4, t_2 t_3, t_2 t_4, t_3 t_4)$$

and the action $T^4 \hookrightarrow \mathbb{C}P^5$ given as the composition of this representation and the standard action of $T^6$ on $\mathbb{C}P^5$. We obtain an effective action $T^3 \hookrightarrow \mathbb{C}P^5$

We prove:

**Theorem**

$\mathbb{C}P^5 / T^3$ is homeomorphic to $\partial \Delta_{4,2} \ast \mathbb{C}P^2 \equiv S^2 \ast \mathbb{C}P^2$. 
The Plücker embedding of $G_{4,2}$ into $\mathbb{C}P^5$ is equivariant under the canonical action of $T^4$ on $G_{4,2}$ and the action of $T^4$ on $\mathbb{C}P^5$ given by representation $T^4 \to T^6$ by the second symmetric power. It implies:

$$G_{4,2}/T^3 \subset \mathbb{C}P^5/T^3 : S^2 \ast \mathbb{C}P^1 \subset S^2 \ast \mathbb{C}P^2,$$

where $\mathbb{C}P^1 \subset \mathbb{C}P^2$

$$(c, 1) \to (c : 1 : (1 - c)), \ (1, 0) \to (0, 0, 1).$$
There is the standard moment map $\mu : G_{4,2} \to \mathbb{R}^4$ defined by:

$$
\mu(X) = \frac{\sum_J |P^J(X)|^2 \delta_J}{\sum_J |P^J(X)|^2},
$$

where $J \subset \{1, 2, 3, 4\}$, $|J| = 2$ and $P^J(X)$ are the Plücker coordinates for $X \in G_{4,2}$ and $\delta_J \in \mathbb{R}^4$ is given by

$$(\delta_J)_i = 1, \ i \in J, \ (\delta_J)_i = 0, \ i \notin J.$$  

This map is invariant for the canonical action of $T^4$ on $G_{4,2}$ and trivial action on $\mathbb{R}^4$.

$\text{Im} \mu = \Delta_{4,2} — \text{octahedron.}$
There exists a smooth atlas \((M_J, \varphi_J)\) for \(G_{4,2}\):

\[
M_J = \{X \in G_{4,2} \mid P^J(X) \neq 0\}, \quad \varphi_J : M_J \to \mathbb{C}^4.
\]

\(X \in M_J \Rightarrow\) it can be represented by matrix \(A\) such that \(A_J = I_d\) and

\[
\varphi_J(X) = (a_{ij}(X)) \in \mathbb{C}^4, \quad i \notin J
\]

Each chart \(M_J\) is \(T^3\)-invariant, everywhere dense set in \(G_{4,2}\) and contains exactly one fixed point which maps to zero by the coordinate map.

For any chart \((M_J, \varphi_J)\) it is given the characteristic homomorphism \(\alpha_J : T^3 \to T^4\) such that the homeomorphism \(\varphi_J\) is \(\alpha_J\) - equivariant:

\[
\varphi_J(tm) = \alpha_J(t)\varphi_J(m), \quad t \in T^3, \quad m_J \in M_J.
\]

For any characteristic homomorphism \(\alpha_J : T^3 \to T^4\), the weight vectors are pairwise linearly independent.

The map \(\mu\) gives the bijection between the set of fixed points and the set of vertices of the polytope \(\Delta_{4,2}\).
We assume the following to be given:

- a smooth, closed simply connected manifold $M^{2n}$;

- a smooth, effective action $\theta$ of the torus $T^k$ on $M^{2n}$, where $2 \leq k \leq n$, such that the stabilizer of any point is connected;

- an open $\theta$-equivariant map $\mu : M^{2n} \rightarrow \mathbb{R}^k$ whose image is a $k$-dimensional convex polytope, where $\mathbb{R}^k$ is considered with trivial $T^k$ - action.

- $\mu$ - we call an *almost moment map*.

- $\text{Im}\mu$ we denote by $P^k$.

- It is defined the characteristic function

$$\chi : M^{2n} \rightarrow S(T^k) \text{ by } \chi(x) = \text{stab}(x)$$

- The function $\chi$ induces mapping from $M^{2n}/T^k \rightarrow S(T^k)$. 
Axiom 1:

There is a smooth atlas $\mathcal{M} = \{(M_i, \varphi_i)\}_{i \in I}$ with the homeomorphisms $\varphi_i : M_i \to \mathbb{R}^{2n} \approx \mathbb{C}^n$ for the fixed identification $\approx$, such that any chart $M_i$

- is $T^k$-invariant,
- contains exactly one fixed point $x_i$ with $\varphi_i(x_i) = (0, \ldots, 0)$,
- the closure of $M_i$ is $M^{2n}$.

Corollary.
The action of $T^k$ on $M^{2n}$ has finitely many isolated fixed points.
Axiom 1:
There is a smooth atlas \( \mathcal{M} = \{(M_i, \varphi_i)\}_{i \in I} \) with the homeomorphisms \( \varphi_i : M_i \to \mathbb{R}^{2n} \cong \mathbb{C}^n \) for the fixed identification \( \cong \), such that any chart \( M_i \)
- is \( T^k \)-invariant,
- contains exactly one fixed point \( x_i \) with \( \varphi_i(x_i) = (0, \ldots, 0) \),
- the closure of \( M_i \) is \( M^{2n} \).

Corollary. The action of \( T^k \) on \( M^{2n} \) has finitely many isolated fixed points.
Denote by $m$ the number of fixed points for $T^k$-action on $M^{2n}$. The charts given by Axiom 1 we enumerate as $(M_1, \varphi_1), \ldots, (M_m, \varphi_m)$. The sets $Y_i = \partial M_i = M - M_i$ are closed and $T^k$-invariant. Define the sets $W_\sigma$, where $\sigma = \{i_1, \ldots, i_l\} \subseteq \{1, \ldots, m\}$ as:

$$W_\sigma = M_{i_1} \cap \cdots M_{i_l} \cap Y_{i_{l+1}} \cap \cdots Y_{i_m},$$

where $\{i_{l+1}, \ldots, i_m\} = \{1, \ldots, m\} - \{i_1, \ldots, i_l\}$.

**Definition**

The non-empty set $W_\sigma$ is called admissible and the corresponding set $\sigma$ is called admissible too.

**Lemma**

The admissible sets $W_\sigma$ are $T^k$-invariant, pairwise disjoint and $M^{2n} = \bigcup W_\sigma$. 
$W_{\{1,\ldots,m\}}$ is an admissible set which is everywhere dense in $M^{2n}$.

$W_{\{i\}}$ is an admissible set for any $1 \leq i \leq m$. 
Axiom 2:
The map $\mu$ gives the bijection between the set of fixed points and the set of vertices of the polytope $P^k$. 

Let $S(P_k)$ be the family of convex polytopes which are spanned by the vertices of the polytope $P_k$ and $\{W_\sigma\}$ the family of all admissible sets.

Define the map $s: \{W_\sigma\} \rightarrow S(P_k)$ by $s(W_\sigma) = P_\sigma$, where $\sigma = \{i_1, \ldots, i_l\}$ and $P_\sigma = \text{convhull}(v_{i_1}, \ldots, v_{i_l})$, and $v_{i_1}, \ldots, v_{i_l}$ are the vertices of the polytope $P_k$ determined by $v_{ij} = \mu(x_{ij})$ for $x_{ij} \in M_{ij} - \text{the fixed point}$.

Definition
A polytope $P_\sigma \in S(P_k)$ is said to be admissible if it corresponds to an admissible set.

The polytope $P_k$ is an admissible polytope, where $\sigma = \{1, \ldots, m\}$. 

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Axiom 2:

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and $v_{i_1}, \ldots, v_{i_l}$ are the vertices of the polytope $P^k$ determined by

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Definition

A polytope $P_\sigma \in S(P^k)$ is said to be admissible if it corresponds to an admissible set.

The polytope $P^k$ is an admissible polytope, where $\sigma = \{1, \ldots, m\}$. 
For a general $(2n, k)$-manifold two admissible polytopes may intersect.

See for example complex flag manifold $F_3$ as $(6, 2)$-manifold and complex Grassmann manifold $G_{4,2}$ as $(8, 3)$-manifold.

**Definition**

A point $p \in P^k$ is said to be singular if $p \in P_{\sigma_1} \cap P_{\sigma_2}$ for some $P_{\sigma_1}, P_{\sigma_2} \in \mathcal{S}$, thus we obtain the set of singular points.
Denote by $\hat{\mu} : M^{2n} / T^k \to P^k$ the map induced by the almost moment map $\mu$.

**Axiom 3:**

The almost moment map $\mu$:

- gives the mapping from $W_\sigma$ to $P_\sigma$, 
- induces fiber bundle $\hat{\mu} : W_\sigma / T^k \to P_\sigma$. 

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Choose $x_\sigma \in \mathring{P}_\sigma$ and let $F_\sigma = \mu^{-1}(x_\sigma)$.

**Definition**

The set $F_\sigma$ we call the set of parameters of the admissible set $W_\sigma$. It is the fiber of the bundle $\mu^\circ : W_\sigma / T^k \rightarrow \mathring{P}_\sigma$.

Since $\mathring{P}_\sigma$ is contractible we obtain:

**Corollary.** The fiber bundle $\mu^\circ : W_\sigma / T^k \rightarrow \mathring{P}_\sigma$ is isomorphic to the trivial bundle. Hence $W_\sigma / T^k$ is homeomorphic to $\mathring{P}_\sigma \times F_\sigma$. 
The boundary $\partial W_\sigma = \overline{W_\sigma} - W_\sigma$ of an admissible set $W_\sigma$ is contained in the union of the admissible sets $W_{\tilde{\sigma}}$ for all subsets $\tilde{\sigma} \subset \sigma$.

In the paper of Gel’fand-Serganova (UMN, 1987) it is given the description of the action of $T^6$ on the Grassmann manifold $G_{7,3}$ from which we deduce the example of our $(2n, k)$-manifold for which

$$\partial W_\sigma \subset \bigcup_{\tilde{\sigma} \subset \sigma} W_{\tilde{\sigma}}.$$
More precisely, consider in $G_{7,3}$ the point given by a matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & b_1 & c_1 & d_1 \\
0 & 1 & 0 & a_2 & 0 & c_2 & d_2 \\
0 & 0 & 1 & a_3 & b_3 & 0 & d_3
\end{pmatrix}
$$

Its $(\mathbb{C}^*)^6$-orbit coincides with a admissible set $W_\sigma$ which contains it. Thus the set $F_\sigma$ of parameters of $W_\sigma$ is a point.

The point given by a matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & b_1 & c_1 & 0 \\
0 & 1 & 0 & a_2 & 0 & c_2 & 0 \\
0 & 0 & 1 & a_3 & b_3 & 0 & 0
\end{pmatrix}
$$

belongs to $\partial W_\sigma$. This point belongs to the admissible set $W_{\sigma'}$ and the set of parameters $F_{\sigma'}$ of $W_{\sigma'}$ is two-dimensional.
For a given trivialization $\xi_\sigma : W_\sigma / T^k \to F_\sigma$ and any point $c_\sigma \in F_\sigma$ the leaf $W_\sigma[\xi_\sigma, c_\sigma] \subseteq W_\sigma$ is defined as

$$W_\sigma[\xi_\sigma, c_\sigma] = (\pi^{-1} \circ \xi^{-1}_\sigma)(c_\sigma),$$

where $\pi : W_\sigma \to W_\sigma / T^k$ is the projection.

**Axiom 4:**

For any admissible $\sigma$ there exists the trivialization $\xi_\sigma : W_\sigma / T^k \to F_\sigma$ such that for any $c_\sigma \in F_\sigma$ the boundary $\partial W_\sigma[\xi_\sigma, c_\sigma]$ of the leaf $W_\sigma[\xi_\sigma, c_\sigma]$ of $W_\sigma$ is the union of the leaves $W_{\bar{\sigma}}[\xi_{\bar{\sigma}}, c_{\bar{\sigma}}]$ for exactly one $c_{\bar{\sigma}} \in F_{\bar{\sigma}}$, where $P_{\bar{\sigma}}$ runs through the admissible faces for $P_\sigma$.

Note. Axiom 4 is motivated by the results of Atiyah, Guillemin-Sternberg and Gel’fand-MacPherson in the case of $(\mathbb{C}^*)^k$-action on $M^{2n}$. 

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Let Axiom 4 is satisfied. Since \( \mu(W_\sigma[\xi_\sigma, c_\sigma]) = P_\sigma \) we obtain:

**Lemma**

For any \( c_\sigma \in F_\sigma \), the boundary \( \partial W_\sigma[\xi_\sigma, c_\sigma] \) of the leaf \( W_\sigma[\xi_\sigma, c_\sigma] \) of the stratum \( W_\sigma \) is the union of the leaves \( W_{\bar{\sigma}}[\xi_{\bar{\sigma}}, c_{\bar{\sigma}}] \) for exactly one \( c_{\bar{\sigma}} \in F_{\bar{\sigma}} \), where \( P_{\bar{\sigma}} \) runs through the all faces for \( P_\sigma \).

**Corollary.** Any face of an admissible polytope is an admissible polytope.

**Corollary.** For any pair \( P_{\sigma'} \subset P_\sigma \) there exists the map \( \xi_{\sigma,\sigma'} : F_\sigma \to F_{\sigma'} \) such that if \( P_{\sigma''} \subset P_{\sigma'} \subset P_\sigma \) then \( \xi_{\sigma'',\sigma'} \circ \xi_{\sigma,\sigma'} = \xi_{\sigma,\sigma''} \).
Let $\mathcal{S}$ denotes the set of admissible polytopes. Define the operator $d$ on $\mathcal{S}$ by

$$dP_\sigma \text{ is disjoint union of all proper faces of } P_\sigma.$$  

We obtain CW complex $CW(M^{2n}, P^k)$: the vertices of this complex are the vertices of $P^k$ and open cells are $\mathring{P}_\sigma$ for $P_\sigma \in \mathcal{S}$. We glue them by induction using the operator $d$.

There is the canonical map $\widehat{\pi}: CW(M^{2n}, P^k) \to P^k$.

For any $P_\sigma \in \mathcal{S}$ there is the cell $\mathring{P}_\sigma'$ in $CW(M^{2n}, P^k)$ such that the map $\widehat{\pi}: \mathring{P}_\sigma' \to \mathring{P}_\sigma$ is a homeomorphism.
We obtain the canonical map \( g : M^{2n} \to CW(M^{2n}, P^k) \) defined by

\[
x \in M^{2n} \Rightarrow \exists! W_\sigma, \ x \in W_\sigma \Rightarrow \mu(x) \in P_\sigma,
\]

\[
\exists! y \in P'_\sigma \subseteq CW(M^{2n}, P^k), \ \hat{\pi}(y) = \mu(x).
\]

**Theorem**

The singular points of \( P^k \) can be resolved that is almost moment map \( \mu : M^{2n} \to P^k \) decomposes as \( \mu = \hat{\pi} \circ g \) for the canonical map \( g : M^{2n} \to CW(M^{2n}, P^k) \).

**Note.** As it is shown we have \( \partial W_\sigma \subset \bigcup_{\tilde{\sigma} \subset \sigma} W_{\tilde{\sigma}} \) for \( G_{7,3} \). Consequently in general case the family \( \{ W_\sigma \} \) does not give \( CW \)-complex. Although there are important examples of \((2n, k)\)-manifolds for which \( CW(M^{2n}, P^k) \) is covered by the \( CW \)-complex of admissible sets.
The orbit space $M^{2n}/T^k$ can be described in terms of $CW(M^{2n}, P^k)$, $F_\sigma$ and $\xi_{\sigma,\sigma'}$:

**Theorem**

$$M^{2n}/T^k = \bigcup P_\sigma \times F_\sigma / \approx,$$

where $(x, f_x) \approx (y, f_y)$ if and only if

1. $x = y \in P_{\sigma'} \subset P_\sigma$,
2. $f_y = \xi_{\sigma,\sigma'}(f_x)$. 
Axiom 5:

- $\chi$ is constant on $W_\sigma$ for any admissible set $W_\sigma$.
- If $W_\sigma' \subset W_\sigma$, then $\chi(W_\sigma) \subset \chi(W_\sigma')$.

We call $W_\sigma$ a stratum if Axiom 5 is satisfied.

Define the function

$$\widehat{\chi}: \mathcal{S} \to S(T^k) \text{ by } \widehat{\chi}(P_\sigma) = \chi(x), \ x \in W_\sigma.$$ 

Then $T_\sigma = T^k/\chi(P_\sigma)$ acts freely on $W_\sigma$.

By construction we have:

If $P_\bar{\sigma}$ is a facet of $P_\sigma$ then $\widehat{\chi}(P_\sigma) \subseteq \widehat{\chi}(P_\bar{\sigma})$. 
Moreover it holds:

**Proposition**

If $P_\sigma \in \mathcal{S}$ and $P_{\bar{\sigma}}$ is a facet of $P_\sigma$ then $\widehat{\chi}(P_\sigma) \subset \widehat{\chi}(P_{\bar{\sigma}})$.

**Theorem**

$\text{Codim } \chi(W_\sigma) = \dim T_\sigma = \dim P_\sigma$ for any stratum $W_\sigma$.

**Corollary.** $\dim W_\sigma[\xi_\sigma, c_\sigma] = 2\dim P_\sigma$ for any admissible polytope $P_\sigma$ and any $c_\sigma \in F_\sigma$

**Corollary.** $\dim F_\sigma$ is even.
Almost standard action

Consider an action $\theta$ of the torus $\mathbb{T}^k$ on $\mathbb{C}^n$ given by a representation $\rho : \mathbb{T}^k \to \mathbb{T}^n$ and the standard action of $\mathbb{T}^n$ on $\mathbb{C}^n$.

**Definition**

An action $\theta$ is called almost standard if:

1. it is effective,
2. the origin is the only fixed point,
3. the stabilizer of any point $x \in \mathbb{C}^n$ is connected.
4. its weight vectors are pairwise linearly independent.

**Remark.** For $k = n$ an almost standard action is isomorphic to the standard one.

- The representation $\rho$ can be written as $\rho = (\rho_1, \ldots, \rho_n)$, where $\rho_i : \mathbb{T}^k \to S^1$, $1 \leq i \leq n$.
- The characters $\rho_i$ can be represented as $\rho_i(t) = e^{2\pi \sqrt{-1} \langle \Lambda_i, t \rangle}$, where $\Lambda_i \in \mathbb{Z}^k$ are the weight vectors for the representation $\rho$. 
We obtain the matrix $V$ be a $(k \times n)$-matrix whose rows are given by the weight vectors $\Lambda_i$.

Denote by $P^J(V)$ the Plücker coordinates of the matrix $V$, where $J \subseteq \{1, \ldots, n\}$ and $|J| = k$.

The matrix $V$ gives the linear map $\mathbb{R}^k \to \mathbb{R}^n$.

Furthermore, for any subset $J \subseteq \{1, \ldots, n\}$, the matrix $V^J$ defined by the vectors $\Lambda_j, j \in J$ gives the linear map $f_J : \mathbb{R}^k \to \mathbb{R}^J$.

**Proposition**

If the map $f_J : \mathbb{R}^k \to \mathbb{R}^J$ is induced by an almost standard action of $\mathbb{T}^k$ on $\mathbb{C}^n$ then the image $f_J(\mathbb{Z}^k)$ is a direct summand in $\mathbb{Z}^J$ for any $J \subseteq \{1, \ldots, n\}$.

**Corollary.** The Plücker coordinates $P^J(V) \in \{-1, 0, 1\}$.

**Corollary.** The weight vectors $\Lambda_i, 1 \leq i \leq n$ are primitive.
Axiom 6:

For any chart \((M_i, \varphi_i)\) it is given the characteristic homomorphism \(\alpha_i : \mathbb{T}^k \to \mathbb{T}^n\) such that:

1. its weight vectors are pairwise linearly independent.
2. the homeomorphism \(\varphi_i\) is \(\alpha_i\) - equivariant:

\[
\varphi_i(tx_i) = \alpha_i(t)\varphi_i(x_i), \quad t \in \mathbb{T}^k, \ x_i \in M_i.
\]

Lemma.

Any characteristic homomorphism \(\alpha_i : \mathbb{T}^k \to \mathbb{T}^n\) gives an almost standard action of \(\mathbb{T}^k\) on \(\mathbb{C}^n\).

Corollary.

The number of fixed points \(m \geq k + 1\).
Consider an action of the algebraic torus \( (\mathbb{C}^*)^k \) on \( \mathbb{C}^n \). It induces the action of the compact torus \( T^k \) on \( \mathbb{C}^n \) given by a representation \( \rho : T^k \rightarrow T^n \) and the standard action of \( T^n \) on \( \mathbb{C}^n \).

**Definition**

An action of \( (\mathbb{C}^*)^k \) on \( \mathbb{C}^n \) we call almost standard action if the induced action of \( T^k \) on \( \mathbb{C}^n \) is almost standard.

**Lemma**

For an almost standard action of \( (\mathbb{C}^*)^k \) on \( \mathbb{C}^n \) it holds

1. any one-dimensional \( (\mathbb{C}^*)^k \)-orbit is one of the coordinate axis,
2. for any codimension one subgroup \( H < T^k \) the fixed point set \( (\mathbb{C}^n)^H \) is either the origin either it is one of the coordinate axis.

For an almost standard action of \( (\mathbb{C}^*)^k \) on \( \mathbb{C}^n \) we have the map from the set of coordinate axis to the set of codimension one subgroups of \( T^k \).
Let $H$ be a codimension one subgroup of $T^k$. Then

$$(M^{2n})^H = \bigcup_{1 \leq i \leq m} M^H_i, \quad S^1 = T^k / H \text{ acts smoothly on } (M^{2n})^H.$$ 

Denote by $X^H$ a connected component of $(M^{2n})^H$. Then $X^H$ is a closed submanifold in $M^{2n}$ and $S^1$ acts smoothly on $X^H$. 

The set of one-dimensional orbits
**Proposition**

- $X^H$ is either a fixed point or it is homeomorphic to the sphere $S^2$ equipped with $S^1$-action with the fixed point $\{x_i, x_j\}$.
- For $X^H \cong S^2$ it holds $X^H - \{x_i, x_j\} \subseteq W_{\{i,j\}}$ and $X^H$ is given by the closure of the preimage of the coordinate axis in the chart $M_i$ as well as the corresponding axis in the chart $M_j$.

**Corollary**

The closure of the set of points in $M^{2n}$ which have one-dimensional orbits is given by the union of $\frac{n \cdot m}{2}$ spheres $S^2$.

**Corollary**

$\mu(X^H) = [v_i, v_j]$

**Note:** If $n$ is odd then the number of fixed points $m$ must be even.
Height function of a \((2n, k)\)-manifold

**Definition**

A linear map \( h : \mathbb{R}^k \to \mathbb{R}, h(x) = \langle x, \nu \rangle \) is said to be the height function for \( T^k \) manifold \( M^{2n} \) if:

1. \( h(v_i) \neq h(v_j) \) for any two vertices \( v_i \) and \( v_j \) of \( P^k \),
2. the composition \( h \circ \mu : M^{2n} \to \mathbb{R} \) is a Morse function whose critical points coincides with the fixed points for \( T^k \)-action on \( M^{2n} \).

**Remark** The condition for \( h \circ \mu \) to be a Morse function does not depend on the vector \( \nu \) in general position.

**Axiom 7**

For a \((2n, k)\)-manifold there is a height function \( h : \mathbb{R}^k \to \mathbb{R} \).
Graph of a \((2n, k)\)-manifold

**Definition**

Graph \(\Gamma(M^{2n}, P^k)\) of \((2n, k)\)-manifold \(M^{2n}\) is a graph given by the vertices and 1-dimensional admissible polytopes of \(P^k\).

It is 1-skeleton of the complex \(CW(M^{2n}, P^k)\).

At any vertex of the graph \(\Gamma(M^{2n}, P^k)\) there are exactly \(n\) edges.

The height function produces the orientation of the graph \(\Gamma(M^{2n}, P^k)\).
The index \( \text{ind}(v) \) of the vertex \( v \) of the graph \( \Gamma(M^{2n}, P^k) \) is the number of edges of \( \Gamma(M^{2n}, P^k) \) incoming into vertex \( v \).

We denote by \( h_q \) the number of vertices of \( \Gamma(M^{2n}, P^k) \) having index \( q \).

**Theorem**

*The number \( h_q \) is equal to \( 2q \)-th Betti number for \( M^{2n} \):*

\[
h_q = b_{2q}(M^{2n}), \quad q = 0, \ldots, n.
\]

The classical Poincare duality theorem gives:

**Corollary**

\[
h_q = h_{n-q}, \quad q = 0, \ldots, n.
\]
Let us consider a graph $\Gamma$ with the set of vertices $V$ and the set of edges $E$.

**Definition**

A graph $\Gamma$ is called $\mathbb{Z}^k$-labeled if it is fixed a mapping $l : E \to \mathbb{Z}^k$.

**Definition**

A mapping $s : V \to \mathbb{Z}[t]$, where $t = (t_1, \ldots, t_k)$ is said to be suitable if $s(v_1) - s(v_2)$ is divisible by the linear form $\langle l(r), t \rangle$ for any edge $r$ which connects the vertices $v_1$ and $v_2$.

**Definition**

$\text{GKM}(\Gamma, l)$ -ring of the labeled graph $(\Gamma, l)$ is the ring of all suitable maps $s : V \to \mathbb{Z}[t]$, with the pointwise multiplication.
Let $E$ denote the set of edges of the graph $\Gamma(M^{2n}, P^k)$. We have the function $\hat{\chi} : E \to S(T^k)$ and

$$\dim \hat{\chi}(e) = k - 1, \ e \in E.$$ 

Let $S^1_e = T^k/\hat{\chi}(e), \ e \in E$.

The projection $\rho_e : T^k \to S^1_e$ is given by the vector $l_e \in \mathbb{Z}^k$.

The labeled graph of $T^k$-manifold $M^{2n}$ is the graph $\Gamma(M^{2n}, P^k)$ together with the labeling of its edges given by $e \to l_e$.

This construction is motivated by the construction of GKM graph.
Let $B(2n, k) = BT^k \times_{T^k} M^{2n}$ be the Borel construction of a $(2n, k)$-manifold. The equivariant cohomology for $M^{2n}$ are defined by

$$H^*_T(M^{2n}, \mathbb{Z}) = H^*(B(2n, k), \mathbb{Z}).$$

**Theorem**

The GKM -ring of the oriented labeled graph $\Gamma(M^{2n}, P^k)$ of a $(2n, k)$-manifold $M^{2n}$ is isomorphic to $H^*_T(M^{2n}, \mathbb{Z})$. 
Let $G_{p,q}$ be a complex Grassmann manifold of $q$-dimensional subspaces in $\mathbb{C}^p$ with canonical action of the compact torus $T^p$. There is the almost moment map $\mu : G_{p,q} \to \Delta_{p,q}$, where $\Delta_{p,q}$ is the hypersimplex of the dimension $p - 1$.

Let $F_p$ be a complex flag manifold of complete flags in $\mathbb{C}^p$ with canonical action of $T^p$. The manifold $F_p$ has the almost moment map $\mu : F_p \to P^{p-1}_e$, where $P^{p-1}_e$ is the permutahedron.

**Theorem**

Any $G_{p,q}$ and $F_p$ are $(2n, k)$-manifolds, where $k = p - 1$. 

Svjetlana Terzić  

The theory of $(2n, k)$-manifolds
Manifold $G_{4,2}$

$$\mu(G_{4,2}) = \Delta_{4,2} - \text{octahedron}$$

$\Delta_{4,2} \subset \mathbb{R}^4$ has 6 vertices and they have two coordinates equal to 1 and two equal to 0.

We list the vertices by the indices of the coordinates equal to 1, that is 12, 13, 14, 23, 24, 34.

The admissible polytopes are:

1. $\Delta_{4,2}$;
2. 6 four-sided pyramid;
3. 3 diagonal squares;
4. any face on the boundary for $\Delta_{4,2}$.  

Svetlana Terzić

The theory of $(2n, k)$-manifolds
\(W_{\Delta,4,2}\) is eight-dimensional stratum with the free action of \(T^3\).

\[
W_{\Delta,4,2}/T^3 \cong \Delta_{4,2} \times F_{\Delta,4,2}
\]

where \(F_{\Delta,4,2} = (\mathbb{C} - \{0, 1\})\).

For the 6-dimensional strata with admissible polytope \(P_\sigma\) all \(F_\sigma\) are the points. We describe \(F_\sigma\) in terms of \(F_{\Delta,4,2}\).

- \(F_\sigma = \{0\}\) for \(\sigma = \{12, 13, 23, 24, 34\}\) or \(\sigma = \{12, 13, 14, 24, 34\}\);
- \(F_\sigma = \{\infty\}\) for \(\sigma = \{12, 14, 23, 24, 34\}\) or \(\sigma = \{12, 13, 14, 23, 34\}\);
- \(F_\sigma = \{1\}\) for \(\sigma = \{13, 14, 23, 24, 34\}\) or \(\sigma = \{12, 13, 14, 23, 24\}\).

The height function \(h : \mathbb{R}^4 \rightarrow \mathbb{R}\) is given by
\[
h(x_1, x_2, x_3, x_4) = x_1 + 2x_2 + 4x_3 + 8x_4.
\]
Embedding of $G_{4,2}$ into $\mathbb{C}P^5$ given by the Plucker coordinates is $T^4$-invariant and it is embedding of $(8, 3)$-manifold into $(10, 3)$-manifold. Almost moment map $G_{4,2} \to \Delta_{4,2}$ decomposes into the composition of this embedding and the almost moment map $\mathbb{C}P^5 \to \Delta_{4,2}$.

Any polytope spanned by some subset of vertices for $\Delta_{4,2}$ is admissible polytope for $(10, 3)$-manifold $\mathbb{C}P^5$. 
Manifold $F_3$

$F_3$ - flag manifold in three-dimensional complex space
It is 6-dimensional manifold with effective action of the compact torus $T^2$.
There is the almost moment map $\mu : F_3 \rightarrow \mathbb{R}^2$.
$\text{Im}\mu = P^2$ is a 6-gon - let us enumerate its vertices as 1, \ldots, 6 anticlockwise.
We obtain $(6, 2)$-manifold and $CW(F_3, P^2)$ has

- 6 vertices: \{1, \ldots, 6\}
- 9 edges: [1, 2], [2, 3], \ldots, [5, 6], [1, 4], [2, 5], [3, 6]
- 4 two dimensional cells: [1, \ldots, 6], [1, 2, 3, 4], [3, 4, 5, 6] and [1, 2, 5, 6].

The vertices of the corresponding graph have the indices: 0, 1, 1, 2, 2, 3.
It follows that the $h$-numbers are: 1, 2, 2, 1, which are the Betti numbers of $F_3$. 