Betti numbers of toric origami manifolds

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Symplectic manifolds

A **symplectic manifold** \((M, \omega)\) is a manifold equipped with a **symplectic form** \(\omega \in \Omega^2(M)\) that is closed \((d\omega = 0)\) and non-degenerate.

**Example**

The unit sphere \(S^2\) in \(\mathbb{R}^3\) is a symplectic manifold with \(\omega = d\theta \wedge dh\).

But for \(n > 1\), \(S^{2n}\) cannot admit a symplectic form.
A *symplectic toric manifold* is a compact connected symplectic manifold \((M^{2n}, \omega)\) equipped with an effective hamiltonian action of an \(n\)-torus \(T^n\) and with a corresponding moment map \(\mu: M \rightarrow \mathbb{R}^n\).

**Delzant’s Theorem**

\[
\begin{align*}
\{\text{compact toric symplectic manifolds}\} & \overset{1:1}{\longleftrightarrow} \{\text{Delzant polytopes}\}, \\
(M, \omega, T^n, \mu) & \overset{1:1}{\longleftrightarrow} \mu(M)
\end{align*}
\]

**Example**

\[h \rightarrow +1 \quad -1\]
Origami

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Origami manifolds

An *origami form* on a $2n$-dim’l manifold $M$ is a closed 2-form $\omega$

- $\omega^n$ vanishes transversally on a submanifold $i: Z \hookrightarrow M$;
- $i^*\omega$ has maximal rank, i.e., $(i^*\omega)^{n-1}$ does not vanish;
- the 1-dimensional kernel on $Z$ is the vertical bundle of an oriented $S^1$ fiber bundle $Z \xrightarrow{\pi} B$ over a compact base $B$.

$(M, \omega)$ is called an *origami manifold* with a fold $Z$. 
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Example

For $n \geq 1$, $(S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}, \omega_{\mathbb{C}^n} \oplus 0)$ is an origami manifold with the fold $Z = S^{2n-1} \subset \mathbb{C}^n \oplus \{0\}$, where $\omega_{\mathbb{C}^n} = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$ and the 1-dimensional kernel on $Z$ is the vertical bundle of the Hopf bundle $\pi: S^{2n-1} \rightarrow \mathbb{C}P^{n-1}$. 
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Toric origami manifolds

The action of a Lie group $G$ on an origami manifold $(M, \omega)$ is Hamiltonian if it admits a moment map $\mu : M \to g^*$ satisfying the conditions:

- $\mu$ collects hamiltonian functions, i.e., $d\langle \mu, X \rangle = \iota_X \# \omega$, $\forall X \in g := \text{Lie}(G)$, where $X \#$ is the vector field generated by $X$;
- $\mu$ is equivariant with respect to the given action of $G$ on $M$ and the coadjoint action of $G$ on the dual vector space $g^*$.

A toric origami manifold is a compact connected origami manifold $(M, \omega)$ equipped with an effective Hamiltonian action of a torus $T$ with $\dim T = \frac{1}{2} \dim M$.

NOTE: If $Z = \emptyset$, a toric origami manifold is a toric symplectic manifold.
Toric origami manifolds

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**NOTE**: If $\mathcal{Z} = \emptyset$, a toric origami manifold is a toric symplectic manifold.
Example 1

\[ T = (S^1)^2 \text{ acts on } (S^4, \omega_{\mathbb{C}^2} \oplus 0) \text{ by } \]

\[ (t_1, t_2) \cdot (z_1, z_2, r) = (t_1 z_1, t_2 z_2, r) \]

with moment map

\[ \mu(z_1, z_2, r) = (|z_1|^2, |z_2|^2). \]

Note that the fold is an equator \( S^3 \cong \{ (z_1, z_2, 0) \in S^4 \}. \)
Note that $S^4 \subset \mathbb{C}^2 \oplus \mathbb{R}$ has an origami form $\omega_{\mathbb{C}^2 \oplus 0}$. The origami form on $S^4$ is invariant under the involution $(z_1, z_2, r) \mapsto -(z_1, z_2, r)$.

Hence, it induces an origami form on $\mathbb{R}P^4 = S^4/\mathbb{Z}_2$ whose fold is $\mathbb{R}P^3 \cong \{[z_1, z_2, 0]\}$.

Furthermore, $\mathbb{R}P^4$ is a toric origami manifold with moment map

$$\mu[z_1, z_2, r] = (|z_1|^2, |z_2|^2).$$
Origami templates

$D_n = \text{the set of all Delzant polytopes in }\mathbb{R}^n$
$E_n = \text{the set of all subsets of }\mathbb{R}^n \text{ which are facets of elements of } D_n$
$G = \text{a graph (need not to be simple, i.e., } \exists \text{ multiple edges and loops)}$

An $n$-dimensional origami template consists of a graph $G$, called the template graph, and a pair of maps $\Psi_V : V \to D_n$ and $\Psi_E : E \to E_n$ such that

1. if $e$ is an edge of $G$ with end vertices $u$ and $v$, then $\Psi_E(e)$ is a facet of a each of the polytopes $\Psi_V(u)$ and $\Psi_V(v)$, and these polytopes coincide near $\Psi_E(e)$; and
2. if $v$ is an end vertex of each of two distinct edges $e$ and $f$, then $\Psi_E(e) \cap \Psi_E(f) = \emptyset$. 
Examples

\[ \Psi_V(u) = \Psi_V(v) = P \]
\[ \Psi_E(e) = F_3 \]

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\[ \Psi_V(u) = P_1 \]
\[ \Psi_V(v) = \Psi_V(w) = P_2 \]
\[ \Psi_E(e) = F_2 \]
\[ \Psi_E(e') = F_6 \]
Theorem [Cannas da Silva-Guillemin-Pires]

There is a one-to-one correspondence

\[
\{\text{toric origami manifolds}\} \leftrightarrow \{\text{origami templates}\},
\]

up to equivariant origami-symplectomorphism on the left-hand side, and
affine equivalence of the image of the template in \(\mathbb{R}^n\) on the right-hand side.
Properties

1. $M$ is orientable $\iff G$ is bipartite.

2. The orbit space $M/T$ is realized as $X = \bigsqcup_{v \in V} (v, \Psi_V(v))/\sim$, where $(u, x) \sim (v, y)$ if there exists an edge $e$ with endpoints $u$ and $v$.

   facets of $X$: $\bigsqcup_{v \in V} (v, F) \sim$

   $F$ facet of $\Psi_V(v)$

   $F$ not a fold facet

3. $X \sim G$. 
Examples

\[ S^4 / T \cong X = \bigcup \]

\[ \Psi_V(u) = \Psi_V(v) = P \]
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\[ \Psi_V(u) = P_1, \Psi_V(v) = P_2 \]
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Motivation

Theorem [Jurkiewicz]

Let $(M, \omega)$ be a symplectic toric manifold corresponding to a Delzant polytope $P$. Then $H^{\text{odd}} = 0$, $b_{2i}(M) = h_i(P)$, and $H^*(M) = \mathbb{Z}(P)/\mathcal{J}$.

Questions

Let $M$ be a toric origami manifold whose origami template is $(G, \Psi_V, \Psi_E)$.

1. Compute the Betti numbers of $M$ by using the orbit space $X$.
2. Describe the cohomology ring of $M$ by using the origami template.
Motivation

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Goal of this talk

[Masuda-Panov, 2006]

If a torus manifold $M$ has a face-acyclic orbit space $M/T$, then $H^{\text{odd}}(M) = 0$ and the even-degree Betti numbers of $M$ can be computed by using the face numbers of the orbit space.

For a toric origami manifold $M$, if the template graph $G$ is a tree, then $M$ is orientable, $M^T \neq \emptyset$, and $M/T$ is face-acyclic. Hence $M$ is a locally standard torus manifold whose orbit space is face-acyclic.

Goal

Let $M$ be an orientable toric origami manifold such that every proper face of $M/T$ is acyclic. Compute the Betti numbers of $M$. 
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Goal

Let $M$ be an orientable toric origami manifold such that every proper face of $M/T$ is acyclic. Compute the Betti numbers of $M$. 
Example

$\Psi_V(u) = P_1, \Psi_V(v) = P_2$

$\Psi_E(e) = F_2$

$\Psi_E(e') = F_6$

$X$ is not acyclic but each proper face is acyclic.

$X$ has a non-acyclic face of codim 1.
Proposition [With Masuda, 2013]

If a toric origami manifold $M$ has a fixed point and the template graph $G$ has no loop, then the quotient map $M \to M/T$ induces an isomorphism $q_* : \pi_1(M) \to \pi_1(M/T)$ and hence $\pi_1(M)$ is a free group.

Corollary

If $G$ is bipartite and every proper face of $M/T$ is acyclic, then

$$H_1(M) = \mathbb{Z}^{b_1(G)},$$

hence $b_1(M) = b_1(G)$. 
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Setting

Let $M$ be an orientable toric origami manifold associated with $(G, \Psi_V, \Psi_E)$ with $b_1(G) > 1$. Choose an edge $e$ in $G$ such that $b_1(G - e) = b_1(G) - 1$.

\[
\begin{array}{ccc}
M & M' & B \\
\updownarrow & \updownarrow & \updownarrow \\
(G, \Psi_V, \Psi_E) & (G - e, \Psi_V, \Psi_E \setminus \{e\}) & \Psi_E(e)
\end{array}
\]
Theorem

Let $M$ be a toric origami manifold such that all proper faces of $M/T$ are acyclic. Then

$$b_{2i+1}(M) = 0, \ 1 \leq i \leq n - 2.$$ 

Moreover, if $M'$ and $B$ are as above, then

$$b_1(M') = b_1(M) - 1,$$
$$b_{2i}(M') = b_{2i}(M) + b_{2i}(B) + b_{2i-1}(B), \ 1 \leq i \leq n - 1.$$
Face numbers of \( M/T \)

Let \( M \) be a toric origami manifold of dim \( 2n \) and \( P \) the simplicial poset dual to \( \partial(M/T) \). We define

\[
f_i = \begin{cases} 
\text{the number of } (n - 1 - i) \text{-faces of } M/T, \\
\text{the number of } i\text{-simplices in } P \text{ for } i = 0, 1, \ldots, n - 1
\end{cases}
\]

and the \( h \)-vector \((h_0, h_1, \ldots, h_n)\) by

\[
\sum_{i=0}^{n-1} f_i (t - 1)^{n-1-i}.
\]

**Lemma**

Assume every proper face of \( M/T \) is acyclic.

\[
f_i(M'/T) = f_i(M/T) + 2f_{i-1}(F) + f_i(F) \text{ for } 0 \leq i \leq n - 1.
\]
Theorem

Let $M$ be a toric origami manifold such that all proper faces of $M/T$ are acyclic. Let $b_j := b_j(M)$. Then

$$
\sum_{i=0}^{n} b_{2i} t^i = \sum_{i=0}^{n} h_i t^i + b_1 (1 + t^n - (1 - t)^n),
$$

in other words, $b_0 = h_0 = 1$ and

$$
b_{2i} = h_i - (-1)^i \binom{n}{i} b_1, \text{ for } 1 \leq i \leq n - 1 \quad b_{2n} = h_n + (1 - (-1)^n)b_1.
$$
Remark

From the previous theorem, we get generalized Dehn-Sommerville relations for $\partial(M/T)$

$$h_{n-i} - h_i = (-1)^i((-1)^n - 1)b_1\binom{n}{i}$$

$$= (-1)^i(\chi(\partial(M/T)) - \chi(S^{n-1}))\binom{n}{i} \text{ for } 0 \leq i \leq n.$$