Mckay correspondence in Quasitoric orbifolds

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An orbifold $X$ is a topological space locally homeomorphic to $\tilde{U}/G$ where $\tilde{U}$ is an open connected set in $\mathbb{R}^n$ and $G$ is a finite group with a smooth action. Orbifolds have atlases and compatibility conditions similar to manifolds. For the notion of orbifold structure, not only the topology of $X$ but the data of group actions is also important.
Orbifold structure

An orbifold structure on a topological space $X$ is given by an atlas of uniformising charts $\{(U, \tilde{U}, G, \phi)\}$ where $U$ is an open set in $X$, $\tilde{U}$ is an open connected subset in $\mathbb{R}^n$, $G$ acts smoothly on $\tilde{U}$ and $\phi : \tilde{U} \to U$ is a continuous map that induces a homeomorphism $\tilde{U}/G \to U$. 
Compatibility of charts

\[ \tilde{U}_1 \xrightarrow{\tilde{W}} \tilde{U}_2 \]

\[ \phi_1 \quad \lambda_1 \]

\[ \quad \text{embedding} \quad \text{embedding} \quad \]

\[ U_1 \xrightarrow{\psi} U_2 \]

\[ U_1 \cap U_2 \]

\[ \quad \text{inclusion} \quad \text{inclusion} \quad \]
**Definition**

The tangent bundle $TU$ of $U$ is defined to be the orbifold $T\tilde{U}/G$ where an element $g \in G$ acts on $T\tilde{U}$ by its differential $dg$. $TX$ is an orbifold with atlas $(TU, T\tilde{U}, G, \psi)$. 
Definition

Let $TX$ be the tangent bundle of the orbifold $X$. An almost complex structure $J$ is a base preserving continuous map $TX \to TX$ satisfying the following:

1. It lifts to a fiberwise linear map $J : T\tilde{U} \to T\tilde{U}$ for every chart $(U, \tilde{U}, G, \phi)$ in the atlas
2. $J^2 | (\text{fiber}) = -I$
3. For every $g \in G$ we have $dgJdg^{-1} = J$
4. It respects embedding of charts.
Definition

Let $X$ be an almost complex orbifold. For a chart $(U, \tilde{U}, G, \phi)$ consider $\bigwedge^n (T^{1,0}(\tilde{U}^*))$ with $G$ action

$$g(dv_1 \wedge \ldots \wedge dv_n) = g^{-1*}(dv_1 \wedge \ldots \wedge dv_n)$$

The canonical bundle $K_X$ is an orbifold with atlas $(|\bigwedge^n (T^{1,0}(\tilde{U}^*))|, \bigwedge^n (T^{1,0}(\tilde{U}^*)), G, \psi_1)$. 
Definition

Let $Y, X$ be complex orbifolds, and $f : Y \to X$ a holomorphic map. The map $f$ is called a blowdown if there exists an analytic subset $X^a$ of $X$ such that

$$f : Y - f^{-1}(X^a) \longrightarrow X - X^a$$

is a biholomorphism, and $f^{-1}$ is meromorphic with singularities along $X^a$. 
Crepant blowdown in complex orbifolds

Definition

A blow down is called crepant if $f^*(K_X) = K_Y$
Projective toric orbifold

**Definition**

A projective toric orbifold $X$ is a simplicial projective toric variety. Since it is simplicial it has orbifold singularity. It has a $\mathbb{C}^n$ action with a dense principal orbit. The $\mathbb{C}^n$ action induces a $U(1)^n$ action.
Definition

The quotient of the $U(1)^n$ action is a $n$-dimensional simple polytope. The quotient map is called $\pi$ and the polytope $P$. An $n$-dimensional polytope is simple if exactly $n$ codimension one faces intersect at each vertex.
Definition

Let $F$ be a codimension one face of $P$. Then $\pi^{-1}(F^\circ)$ has a circle subgroup as its stationary subgroup which is characterized by an integral vector $\lambda(F)$, well defined up to sign (in a suitable integral basis). This integral vector $\pm \lambda(F)$ is normal to $F$. We call a codimension one face $F$ a facet and $\lambda(F)$ the characteristic vector associated to the facet. We have a map $\lambda$ from facets to $\mathbb{Z}^n$. 
Batyrev and Dais proved that string theoretic Hodge numbers are preserved for crepant blowdowns of Gorenstein complex algebraic varieties with toroidal quotient singularities. In general, McKay correspondence for arbitrary compact complex algebraic orbifolds was proved by Lupercio-Poddar and Yasuda.
In easy language quotient singularities are orbifold singularities and Gorenstein means that each $dg$ belongs to $SL(n, C)$. From now on we call such orbifolds $SL$. 
Since projective toric orbifolds have pure Hodge structure the above amounts to Hodge numbers and Betti numbers of Chen-Ruan cohomology being preserved under crepant blowdowns.
The proof requires manipulation of something called E-polynomial which is defined as follows.

\[ e^{p,q}(X) = \sum_{k \geq 0} (-1)^k h^{p,q}(H^k_c(X)). \]  

(1)

The E-polynomial is

\[ E(X; u, v) := \sum_{p,q} e^{p,q}(X) u^p v^q. \]  

(2)
The involvement of Hodge structure restricts the domain of the result to the complex analytic category.
Quasitoric orbifolds were defined by Davis-Januszkiewicz and Poddar-Sarkar by generalizing the projective toric orbifolds. They relaxed the condition that the characteristic vectors have to be normal to the facets. They require the condition that the characteristic vectors of the facets meeting at a vertex have to be linearly independent.
1. Recall: The circle subgroup generated by a characteristic vector fixes the suborbifold represented by the facet.
2. The sign of the characteristic vector defines an orientation of the normal bundle of the suborbifold. These are called omniorientations defined by the characteristic vector.
3. Given a choice of signs of characteristic vector we get an omnioriented Quasitoric orbifold.
\[ F_1 \]
point blowup in $2n$-manifold

2n MANIFOLD POINT BLOW UP
Topologically in a quasitoric manifold blowup will correspond to replacing an torus invariant submanifold of $X$ by the projectivization of its normal bundle. In orbifolds we give a combinatorial description. Combinatorially we replace a face by a facet with a new characteristic vector. Let $Y$ be the space obtained by this operation. Blowdown is the natural map $\rho : Y \rightarrow X$. 
$F_0$ 

blow down 

$F$
Let $F$ be the blowed up codimension $k$ face. If it has a characteristic set $\lambda_F = \{\lambda_1, \lambda_2 \ldots \lambda_k\}$. Let $F_0$ be the new facet after blow up. The characteristic vector $\lambda_0$ is given by

$$\lambda_0 = \sum_{j=1}^{k} b_j \lambda_j, \ b_j \in Q \cap (0, 1) \ \forall j.$$  \ \ (3)
A blow up or blowdown is crepant if

$$\sum_{j=1}^{k} b_j = 1$$

(4)

For almost complex quasitoric orbifolds with some more conditions the two definitions are same.
Chen- Ruan cohomology

Definition

Let $F$ be any codimension $k$ face and $\lambda_F$ be its characteristic set and $N(F)$ be the sub module of $\mathbb{Z}^n$ generated by the the vectors of $\lambda_F$. Define

$$G_F = \sum_{j=1}^{k} a_j \lambda_j \cap \mathbb{Z}^n / N(F)$$ (5)

Here $a_j \in \mathbb{Q} \cap [0, 1)$. $G_F$ is a group.
For $g \in G_F$, define the degree shifting number or age of $g$ to be

$$\iota(g) = \sum a_j.$$  \hspace{1cm} (6)
We write $F \leq H$ if $F$ is a sub-face of $H$, and $F < H$ if it is a proper sub-face. If $F \leq H$ we have a natural inclusion of $G_H$ into $G_F$ induced by the inclusion of $N(H)$ into $N(F)$. Therefore we may regard $G_H$ as a subgroup of $G_F$.

Define the set

$$G_F^\circ = G_F - \bigcup_{F < H} G_H$$  (7)
We define the Chen-Ruan orbifold cohomology of an omnioriented quasitoric orbifold $X$ to be

$$H^*_{CR}(X, R) = \bigoplus_{F \leq P} \bigoplus_{g \in G^o_F} H^{*-2\iota(g)}(X(F), R).$$
An almost complex orbifold is $SL$ if each $dg$ is in $SL(n, C)$. This is equivalent to $\iota(g)$ being integral for every $g$ in $G_F$. Therefore, to suit our purposes, we make the following definition.

\textbf{Definition}

A quasitoric orbifold is said to be quasi-$SL$ if the age of every $g$ in $G_F$ $\iota(g)$ is an integer.
**Theorem**

*For any crepant blowdown $\rho : Y \to X$ quasi-SL orbifolds, Betti numbers of Chen-Ruan cohomology of $X$ and $Y$ are the same.*
idea of Proof

1. Singularities of a quasitoric orbifold are same as that of a projective toric orbifold. In quasitoric orbifold things don’t patch analytically: So obstructions are global.

2. To handle global obstructions we use the fact that any simple polytope has combinatorially equivalent rational or integral realization.

3. From integral polytope we get projective toric orbifolds where the result holds.

4. Since singular or de Rham cohomology of of quasitoric orbifold depend on the combinatorial class of the polytope and since Chen-Ruan cohomology depends on singular cohomology of quasitoric suborbifolds, we can adapt Batyrev-Dais style of proof to our situation.