The Cohomology Algebra of Polyhedral Product Spaces

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The cohomology group

For two graded groups $A^*_\Lambda = \bigoplus_{\alpha \in \Lambda} A^*_\alpha$ and $B^*_\Lambda = \bigoplus_{\alpha \in \Lambda} B^*_\alpha$, their diagonal tensor product group with respect to the index set $\Lambda$ is

$$A^*_\Lambda \otimes_\Lambda B^*_\Lambda = \bigoplus_{\alpha \in \Lambda} A^*_\alpha \otimes B^*_\alpha.$$ 

The usual tensor product is

$$A^*_\Lambda \otimes B^*_\Lambda = \bigoplus_{\alpha, \beta \in \Lambda} A^*_\alpha \otimes B^*_\beta.$$
The cohomology group

For a simplicial complex $K$ with vertex set a subset of $[m]$ and a sequence of CW-complex pairs $(X, A) = \{(X_k, A_k)\}_{k=1}^m$, the polyhedral product space $Z(K; X, A)$ is the subspace of $X_1 \times \cdots \times X_m$ defined as follows. For a subset $\sigma$ of $[m]$, define

$$D(\sigma) = Y_1 \times \cdots \times Y_m, \quad Y_k = \begin{cases} X_k & \text{if } k \in \sigma, \\ A_k & \text{if } k \notin \sigma. \end{cases}$$

Then $Z(K; X, A) = \bigcup_{\sigma \in K} D(\sigma)$. 
The cohomology group

Let $\mathcal{Z}(K; X, A)$ be a polyhedral product space such that every ker $i^*_k$, coker $i^*_k$, im $i^*_k$ are free modules over $\mathbb{F}$, where

$$i^*_k: H^*(X_k; \mathbb{F}) \rightarrow H^*(A_k; \mathbb{F})$$

is the singular cohomology homomorphism induced by the inclusion map. The cohomology group of the polyhedral product space is

$$H^*(\mathcal{Z}(K; X, A); \mathbb{F}) \cong H^*_{\gamma_m}(K) \otimes_{\gamma_m} H^*_{\gamma_m}(X, A).$$
The index set $\Upsilon_m = \{ (\sigma, \omega) \mid \sigma, \omega \subset [m], \sigma \cap \omega = \emptyset \}$.

The total cohomology group of $K$ over $\mathbb{F}$ is

$$H^*_{\Upsilon_m}(K) = \bigoplus_{(\sigma, \omega) \in \Upsilon_m} H^*_{\sigma, \omega}(K),$$

where $H^*_{\sigma, \omega}(K) = \tilde{H}^{*-1}(K_{\sigma, \omega}; \mathbb{F})$, the singular cohomology with degree uplifted by 1 and

$$K_{\sigma, \omega} = \text{link}_K \sigma|\omega = \{ \tau \mid \tau \subset \omega, \sigma \cup \tau \in K, \sigma \cap \tau = \emptyset \}$$

if $\sigma \in K$ and $K_{\sigma, \omega} = \{ \}$ if $\sigma \notin K$. 
The cohomology group of \((X, A)\) over \(\mathbb{F}\) is

\[
H^*_{\mathcal{Y}_m}(X, A) = \bigoplus_{(\sigma, \omega) \in \mathcal{Y}_m} H^*_{\sigma, \omega}(X, A),
\]

where \(H^*_{\sigma, \omega}(X, A) = H^1 \otimes \cdots \otimes H^m\) with

\[
H^k = \begin{cases} 
\ker i^*_k & \text{if } k \in \sigma, \\
\coker i^*_k & \text{if } k \in \omega, \\
\text{im} i^*_k \cong \text{coim} i^*_k & \text{otherwise}.
\end{cases}
\]
The cohomology algebra

An algebra \((A^*, \Pi)\) is a graded group \(A^*\) with product

\[
\Pi: A^* \otimes A^* \rightarrow A^*
\]

a graded group homomorphism. \(\Pi\) may not be associative.

If an algebra \((A^*_\Lambda, \Pi)\) satisfies \(A^*_\Lambda = \bigoplus_{\alpha \in \Lambda} A^*_\alpha\), then the product \(\Pi\) is determined by all its restriction product

\[
\Pi_{\alpha, \beta, \gamma}: A^*_\beta \otimes A^*_\gamma \xrightarrow{i} A^*_\Lambda \otimes A^*_\Lambda \xrightarrow{\Pi} A^*_\Lambda \xrightarrow{p} A^*_\alpha,
\]

where \(i\) is the inclusion and \(p\) is the projection, since for \(b \in A^*_\beta\) and \(c \in A^*_\gamma\),

\[
\Pi(b \otimes c) = \Sigma_{\alpha \in \Lambda} \Pi_{\alpha, \beta, \gamma}(b \otimes c) \text{ with } \Pi_{\alpha, \beta, \gamma}(b \otimes c) \in A^*_\alpha.
\]
The cohomology algebra

For algebras \((A^*_\Lambda = \bigoplus_{\alpha \in \Lambda} A_\alpha, \Pi_1)\) and \((B^*_\Lambda = \bigoplus_{\alpha \in \Lambda} B_\alpha, \Pi_2)\), their diagonal tensor product algebra \((A^*_\Lambda \otimes_\Lambda B^*_\Lambda, \Pi_1 \otimes_\Lambda \Pi_2)\) with respect to \(\Lambda\) is defined as follows. For

\[
a'_\beta \in A^*_\beta, \ a''_\gamma \in A^*_\gamma \text{ with } \Pi_1(a'_\beta \otimes a''_\gamma) = \Sigma_{\alpha \in \Lambda} a_\alpha, \ a_\alpha \in A^*_\alpha,
\]

\[
b'_\beta \in B^*_\beta, \ b''_\gamma \in B^*_\gamma \text{ with } \Pi_2(b'_\beta \otimes b''_\gamma) = \Sigma_{\alpha \in \Lambda} b_\alpha, \ b_\alpha \in B^*_\alpha,
\]

\[
(\Pi_1 \otimes_\Lambda \Pi_2)\left((a'_\beta \otimes b'_\beta) \otimes (a''_\gamma \otimes b''_\gamma)\right) = (-1)^{|a''_\gamma||b'_\beta|} \Sigma_{\alpha \in \Lambda} a_\alpha \otimes b_\alpha.
\]

Equivalently, the restriction products of the three algebras satisfy

\[
(\Pi_1 \otimes_\Lambda \Pi_2)_{\alpha'}^{\beta,\gamma} = (\Pi_1)_{\alpha'}^{\beta,\gamma} \otimes (\Pi_2)_{\alpha'}^{\beta,\gamma}.
\]
The cohomology algebra

Let $\mathcal{Z}(K; X, A)$ be a polyhedral product space such that every $\ker i_k^*$, $\coker i_k^*$, $\text{im} i_k^*$ are free modules over $\mathbb{F}$, the cohomology algebra of the polyhedral product space is

$$(H^*(\mathcal{Z}(K; X, A); \mathbb{F}), \cup)$$

$$\cong (H_{\Upsilon_m}^*(K) \otimes_{\Upsilon_m} H_{\Upsilon_m}^*(X, A), \cup K \otimes_{\Upsilon_m} \Pi(X, A)).$$
The universal cohomology algebra \((H^*_\gamma_m(K), \cup_K)\) is defined as follows. The restriction product

\[\bigcup^R_K : H^*_\sigma,\omega'(K) \otimes H^*_\sigma'',\omega''(K) \to H^*_\sigma,\omega(K)\]

of \(\cup_K\) is induced by the cochain complex homomorphism

\[\Pi^R_K : \tilde{C}^*(K_{\sigma'},\omega') \otimes \tilde{C}^*(K_{\sigma''},\omega'') \to \tilde{C}^*(K_{\sigma},\omega)\]

defined as follows.
The cohomology algebra

(1) $\Pi^R_K = 0$ if $\sigma' \cup \sigma'' \not\subset \sigma$ or $\omega \not\subset \omega' \cup \omega''$.

(2) For $\{i_1, \ldots, i_s\} \in \widehat{C}^*(K_{\sigma'}, \omega')$, $\{j_1, \ldots, j_t\} \in \widehat{C}^*(K_{\sigma''}, \omega'')$ and $\sigma' \cup \sigma'' \subset \sigma$, $\omega \subset \omega' \cup \omega''$,

$$\Pi^R_K(\{i_1, \ldots, i_s\} \otimes \{j_1, \ldots, j_t\}) = 0$$

if $\{i_1, \ldots, i_s\} \cap \{j_1, \ldots, j_t\} \neq \emptyset$ or $\{j_1, \ldots, j_t\} \cap \omega' \neq \emptyset$, and otherwise,

$$\Pi^R_K(\{i_1, \ldots, i_s\} \otimes \{j_1, \ldots, j_t\}) = (-1)^\tau \{k_1, \ldots, k_u\},$$

where $\{k_1, \ldots, k_u\} = \{i_1, \ldots, i_s\} \cup \{j_1, \ldots, j_t\}$ and $(-1)^\tau$ is the sign of the permutation $\begin{pmatrix} i_1 & \ldots & i_s & j_1 & \ldots & j_t \\ k_1 & \ldots & k_s & k_{s+1} & \ldots & k_u \end{pmatrix}$ and $s, t, u$ may be 0.
The cohomology algebra of \((X, A)\) is
\[
(H^\_\ast \left( X \_1, A \_1 \right) \otimes \cdots \otimes H^\_\ast \left( X \_m, A \_m \right), \Pi \_1 \otimes \cdots \otimes \Pi \_m)
\]
with \(\Pi \_k\) on \(H^\_\ast \left( X \_k, A \_k \right) = \ker i^\_k \oplus \coker i^\_k \oplus \im i^\_k\) defined as follows.

1. \(\Pi \_k \left( x \otimes y \right) = \Pi \_X \_k \left( x \otimes y \right)\) for all \(x, y \in \ker i^\_k \oplus \im i^\_k = \ker i^\_k \oplus \coim i^\_k = H^\ast \left( X \_k \right)\),
2. \(\Pi \_k \left( a \otimes b \right) = \Pi \_A \_k \left( a \otimes b \right)\) for all \(a, b \in H^\ast \left( A \_k \right)\) such that \(a\) or \(b\) is in \(\coker i^\_k\),
3. \(\Pi \_k \left( x \otimes a \right) = \Pi \_k \left( a \otimes x \right) = 0\) for all \(x \in \ker i^\_k\) and \(a \in \coker i^\_k\).
The applications

The normal algebra \((H^*_{\widehat{\text{ym}}}(K), \widehat{\cup}_K)\) of \(K\) is defined as follows. The restriction product

\[
H^*_{\sigma', \omega'}(K) \otimes H^*_{\sigma'', \omega''}(K) \rightarrow H^*_\sigma, \omega(K)
\]

of \(\widehat{\cup}_K\) coincides with the restriction product of the universal product \(\cup_K\) if \(\sigma' \cup \sigma'' = \sigma\) and \(\omega = \omega' \cup \omega''\) and all other restriction products of \(\widehat{\cup}_K\) is 0.

The special algebra \((H^*_{\text{ym}}(K), \cup_K)\) of \(K\) is defined as follows. The restriction product

\[
H^*_{\sigma', \omega'}(K) \otimes H^*_{\sigma'', \omega''}(K) \rightarrow H^*_\sigma, \omega(K)
\]

of \(\cup_K\) coincides with the restriction product of the universal product \(\cup_K\) if \(\sigma' \sqcup \sigma'' = \sigma\) and \(\omega = \omega' \sqcup \omega''\) and all other restriction products of \(\widehat{\cup}_K\) is 0.
The applications

Suppose for \( k = 1, \ldots, m \), \( H^*_\Upsilon(X_k, A_k) \) is a free module such that \( \text{coker} \ i^*_k \) is an ideal of \( H^*(A_k) \) and \( \text{coim} \ i^*_k = \text{im} \ i^*_k \) is a subalgebra of both \( H^*(A_k) \) and \( H^*(X_k) \). So \( (H^*_\Upsilon m (X, A), \Pi_{(X, A)}) \) is an associative, commutative algebra with unit. Then

\[
(H^*(\mathcal{Z}(K; X, A)), \cup) \cong (H^*_{\Upsilon m} (K) \otimes_{\Upsilon m} H^*_\Upsilon m (X, A), \overline{\cup}_K \otimes_{\Upsilon m} \Pi_{(X, A)}).
\]

Specifically,

\[
(H^*(\mathcal{Z}(K; SX, SA)), \cup) \cong (H^*_{\Upsilon m} (K) \otimes_{\Upsilon m} H^*_\Upsilon m (SX, SA), \overline{\cup}_K \otimes_{\Upsilon m} \Pi_{(X, A)}).
\]
The applications

Let $\mathcal{L}_m = \{ (\sigma, \emptyset) \in \Upsilon_m \}$, $\mathcal{R}_m = \{ (\emptyset, \omega) \in \Upsilon_m \}$.

The left universal algebra $(H^*_\mathcal{L}_m(K), \cup_K)$ of $K$ is the subalgebra of $(H^*_\Upsilon_m(K), \cup_K)$ with $H^*_\mathcal{L}_m(K) = \bigoplus_{(\sigma, \emptyset) \in \mathcal{L}_m} H^*_\sigma,\emptyset(K)$.

The right universal algebra $(H^*_\mathcal{R}_m(K), \cup_K)$ of $K$ is the quotient algebra of $(H^*_\Upsilon_m(K), \cup_K)$ over the ideal $\bigoplus_{(\sigma, \omega) \in \Upsilon_m, \sigma \neq \emptyset} H^*_\sigma,\omega(K)$.

When the vertex set of $K$ is $[m]$, by Hochster theorem,

$$H^*_\mathcal{R}_m(K) \cong \bigoplus_{\omega \subseteq [m]} H^*(K|\omega) \cong \text{Tor}^*_\mathbb{F}[x_1, \ldots, x_m](\mathbb{F}(K), \mathbb{F}),$$

where $\mathbb{F}(K)$ is the Stanley-Reisner face ring of $K$. 
Suppose $H^*_\Upsilon(X_k, A_k)$ is a free module.

The left cohomology algebra $H^*_{L_m}(X, A)$ of $(X, A)$ is the subalgebra of $H^*_\Upsilon(X, A)$ with

$$H^*_{L_m}(X, A) = \bigoplus_{(\sigma, \emptyset) \in L_m} H^*_{\sigma, \emptyset}(X, A).$$

The right cohomology algebra $H^*_{R_m}(X, A)$ of $(X, A)$ is the quotient algebra of $H^*_\Upsilon(X, A)$ modular the ideal

$$\bigoplus_{(\sigma, \omega) \in \Upsilon_m, \sigma \neq \emptyset} H^*_{\sigma, \omega}(X, A).$$
The applications

If every $i_k^*: H^*(X_k) \to H^*(A_k)$ is an epimorphism between free modules, then

$$\left( H^*(Z(K; X, A)), \cup \right) \cong \left( H^*_{Lm}(X, A), \Pi_{(X,A)} \right).$$

Specifically, if $(X_k, A_k) = (CP^\infty, *)$ for all $k$, then

$$H^*_{Lm}(X, A) \cong \mathbb{F}(K).$$

If every $i_k^*: H^*(X_k) \to H^*(A_k)$ is a monomorphism between free modules, then

$$\left( H^*(Z(K; X, A)), \cup \right) \cong \left( H^*_m(K) \otimes R_m H^*(A_1 \times \cdots \times A_m), \cup K \otimes R_m \cup \right).$$