A family of polytopal moment-angle manifolds

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Moment-angle manifolds

A moment-angle manifold is a moment-angle complex \((D^2, S^1)^K\) \((K \subset 2^{[m]}, [m] := \{1, 2, \ldots, m\})\) that is homeomorphic to a manifold:

- \((D^2, S^1)^K\) is homeomorphic to a topological manifold if \(|K|\) is homeomorphic to a sphere (Buchstaber-Panov);
- it is homeomorphic to a smooth manifold, if \(|K| = \partial P^*\) where \(P\) is a simple convex polytope (Buchstaber-Panov, Bosio-Meersseman), or more generally, if \(K\) is the underlying complex of a complete simplicial fan (Panov-Ustinovsky).
The link model for polytopal MACs (Bosio-Meersseman)

- Note that \((D^2, S^1)^K\) can be embedded in \(\mathbb{C}^m = \mathbb{R}^{2m}\).
- Realize \(|K| \subset \mathbb{R}^n\) as the boundary complex of a convex polytope \(P^*\), then the Gale transform of \(\text{Vert}(|K|)\) gives rise to a configuration \(A_K = (A_1, A_2, \ldots, A_m)\) with \(A_i \in \mathbb{R}^d, d = m - n - 1\). Then

\[
X_{A_K} := \{(z_i)_{i=1}^m \in \mathbb{C}^m \mid \begin{cases} \sum_{i=1}^m A_i|z_i|^2 = 0 \\ \sum_{i=1}^m |z_i|^2 = 1 \end{cases} \}
\]

is a transverse intersection of \(d\) quadrics with the unit Euclidean sphere in \(\mathbb{C}^m\) (Bosio-Meersseman, \ldots).
The natural actions of $\mathbb{T}^m$ on $X_{AK}$ and $(D^2, S^1)^K$ give homeomorphic quotient spaces, as manifold with corners:

$$
\begin{align*}
\sum_{i=1}^m A_i|z_i|^2 &= 0, \\
\sum_{i=1}^m |z_i|^2 &= 1,
\end{align*}
\quad\pi \quad P' := \begin{cases}
\sum_{i=1}^m A_i r_i = 0, \\
\sum_{i=1}^m r_i = 1,
\end{cases} \quad r_i \geq 0,
$$

where $P' = P$ combinatorially (with $m$ facets). Hence topologically

$$
\begin{array}{c}
X_{AK} \\ \uparrow \\
X_{AK} \cap \mathbb{R}^m
\end{array}
\quad\begin{array}{c}
\mathcal{U} (\mathbb{T}^m, P') \\ \uparrow \\
\mathcal{U} ((\mathbb{Z}_2)^m, P')
\end{array}
\quad\begin{array}{c}
\mathcal{U} \quad (D^2, S^1)^K \\ \uparrow \\
\mathcal{U} \quad (D^1, S^0)^K
\end{array}
$$

where $\mathcal{U} (\mathbb{T}^m, P')$ and $\mathcal{U} ((\mathbb{Z}_2)^m, P')$ are respective Basic constructions (Davis-Januszkiewicz).
An example

\[ m = 2 \quad K = \{1, 2\} \quad (D^1, S^0)^K \max\{|x|, |y|\} = 1 \]

\[ X_{AK} \cap \mathbb{R}^2 \quad x^2 + y^2 = 1 \]

\[ U \left((\mathbb{Z}_2)^2, P'\right) \quad |x| + |y| = 1 \]
An example

\[ X_{\mathcal{A}_K} \cap \mathbb{R}^2 \quad \rightarrow \quad \mathcal{U} \left( (\mathbb{Z}_2)^2, P' \right) \ni (\text{sgn}(x)x^2, \text{sgn}(y)y^2) \]

is not PD.
An example

Why not radial projections?
An $m$-tuple $A = (A_i)_{i=1}^m$ with $A_i \in \mathbb{R}^d$ ($m > d$) is admissible, if it satisfies

- (Siegel condition) $0 \in \text{conv}(A_i)_{i=1}^m$, and
- (Weak hyperbolicity condition) $0 \in \text{conv}(A_i)_{i \in I} (I \subset [m]) \Rightarrow \text{card}(I) > d$.

In the smooth foliation $\mathcal{F}_A$ given by

$$(z, T) \in \mathbb{C}^m \times \mathbb{R}^d \mapsto (z_i e^{\langle A_i, T \rangle})_{i=1}^m \in \mathbb{C}^m \quad z = (z_i)_{i=1}^m,$$

a leaf $L_z$ passing through $z$ (i.e. the orbit $F(z, -)$) is called a Siegel leaf if $0 \notin \overline{L_z}$, otherwise $L_z$ is called a Poincaré leaf.
Example: Siegel leaves

\[ F(x, y, t) = (e^t x, e^{-t} y) \]
Several beautiful theorems

Let $S_A \subset \mathbb{C}^m$ be the union of all Siegel leaves of $\mathcal{F}_A$, and for any $z \in \mathbb{C}^m$, define

$$I_z := \{ i \in [m] \mid |z_i| > 0 \}.$$

**Theorem (Meersseman-Verjovsky-Bosio)**

$A = (A_i)_{i=1}^m$ is admissible iff the intersection of the $d$ quadrics $
\sum_{i=1}^m A_i |z_i|^2 = 0$ with the sphere $\sum_{i=1}^m |z_i|^2 = 1$ is non-empty and transverse.

**Theorem (Camacho-Kuiper-Palls-Meersseman-Verjovsky-Bosio)**

If $A$ is admissible, then we have

$$S_A = \{ z \in \mathbb{C}^m \mid 0 \in \text{conv}(A_i)_{i \in I_z} \}.$$
Several beautiful theorems

Let $p \geq 1$ be a real number, and let

$$X_A(p) := \{ z = (z_i)_{i=1}^m \in \mathbb{C}^m \mid \begin{cases} \sum_{i=1}^m A_i |z_i|^p = 0 \\ \|z\|_p = 1 \end{cases} \},$$

where $\|z\|_p = \left( \sum_{i=1}^m |z_i|^p \right)^{\frac{1}{p}}$.

Theorem (Bosio-Meersseman)

$\mathcal{F}_A|_{S_A}$ is trivial:

$$\Psi_A(2): (z, T, r) \in X_A(2) \times \mathbb{R}^d \times \mathbb{R}_{>0} \mapsto r(e^{\langle A_i, T \rangle} z_i)_{i=1}^m \in S_A$$

is a global diffeomorphism.
The three models

Siegel leaves

Main results

Applications

When \( p \) varies

\[ \Psi_A(p): (z, T, r) \in X_A(p) \times \mathbb{R}^d \times \mathbb{R}_{>0} \mapsto r(e^{\langle A_i, T \rangle} z_i)_{i=1}^m \in S_A \]

is a homeomorphism, for every \( p \geq 1 \). Moreover,

\[ \pi_1 \circ \Psi_A^{-1}(p): S_A \to X_A(p) \hookrightarrow S_A \]

is continuous in \( p \in (1, \infty] \) with each \( z \in S_A \) given, by which each leaf intersects \( X_A(2) \) and \( X_A(p) \) once respectively, yielding a homeomorphism between them.

How about

\[ p \to \infty? \]
The three models

Siegel leaves

Main results

Applications

MAC as the limit set

If $A$ is admissible, then

$$K_A := \{\sigma \subset [m] \mid 0 \in \text{conv}(A_i)_{i \in [m] \setminus \sigma}\}$$

is an abstract simplicial complex. Set $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and $S^1 = \partial D^2$, then $\left(\|z\|_\infty = \max\{|z_i|\}_{i=1}^m\right)$

$$(D^2, S^1)^K_A := \bigcup_{\sigma \in K_A} D(\sigma) \quad D(\sigma) = \prod_{i=1}^m Y_i; \quad Y_i = \begin{cases} D^2 & \text{if } i \in \sigma, \\ S^1 & \text{otherwise}. \end{cases}$$

Proposition

For each $z_0 \in S_A$, any point $z' \in \{z \in \mathbb{C}^m \mid \|z\|_\infty = 1\}$ contained in the closure of the path $\pi_1 \circ \Psi_A^{-1}(p)(z_0)$ (as a function of $p$) lies in $(D^2, S^1)^K_A$. 
When $A$ is centered at the origin

It is likely that each leaf $L_z$ indeed intersects $(D^2, S^1)^{K_A}$, at only one point. This is the case if $A = (A_i)_{i=1}^m$ is centered at the origin:

$$\sum_{i=1}^m A_i = 0.$$

**Theorem**

If $A$ is admissible and centered at the origin, then

$$\Psi_A(\infty): (z, T, r) \in (D^2, S^1)^{K_A} \times \mathbb{R}^d \times \mathbb{R}_{>0} \mapsto r (e^{\langle A_i, T \rangle} z_i)_{i=1}^m \in S_A$$

is a homeomorphism. Moreover,

$$\pi_1 \circ \Psi_A^{-1}(2)|_{(D^2, S^1)^{K_A}}: (D^2, S^1)^{K_A} \rightarrow X_A(2)$$

is a homeomorphism with the inverse $\pi_1 \circ \Psi_A^{-1}(\infty)|_{X_A(2)}$. 
A rigidity theorem revisited

Theorem (Bosio-Meersseman)

If two \( m \)-tuples \( A, A' \) are both admissible and centered at the origin, such that \( K_A = K_{A'} \) simplicially, then \( X_A(2) \) is \( \mathbb{T}^m \)-equivariantly diffeomorphic to \( X_{A'}(2) \).

- Note that in the “real” case, this is proved by Davis, using the language of (Coxeter) orbifolds.
- For general \( A \) (admissible), there exists a smooth family of \( m \)-tuples \( A_t \), with \( t \in [0, 1] \), such that a) \( A_0 = A \), b) \( A_1 \) is centered at the origin and c) \( A_t \) is admissible, for each \( t \).
A rigidity theorem revisited

Proof.

Extend the simplicial isomorphism $K_A \to K_{A'}$ to a permutation $\phi : [m] \to [m]$, then consider the diagram

\[
\begin{array}{ccc}
S_A & \xrightarrow{\phi \text{\ diffeo.}} & S_{A'} \\
\uparrow & & \downarrow \\
X_A(2) & \rightarrow & X_{A'}(2) \\
\pi_1 \circ \Psi_A^{-1}(\infty) & \text{homeo.} & \pi_1 \circ \Psi_{A'}^{-1}(2) \text{ homeo.} \\
(D^2, S^1)^K_A & \xrightarrow{\phi \text{\ homeo.}} & (D^2, S^1)^{K_{A'}}.
\end{array}
\]
The three models
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Main results
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Whitehead triangulations of $X_A(2) \cap \mathbb{R}^m$

**Theorem**

*Suppose $A$ is admissible and centered at the origin, then the restriction of the smooth retraction $\pi_1 \circ \Psi_A^{-1}(2): S_A \cap \mathbb{R}^m \rightarrow X_A(2) \cap \mathbb{R}^m$ to either $(D^1, S^0)^{K_A}$ or $X_A(1) \cap \mathbb{R}^m$ induces a PD homeomorphism onto $X_A(2) \cap \mathbb{R}^m$.>*

Therefore by a theorem of Whitehead, the smooth structure on $X_A(2) \cap \mathbb{R}^m$ uniquely determines a PL structure on $(D^1, S^0)^{K_A}$; if $(D^1, S^0)^{K_A}$ has another PL structure, then it is either

- non-smoothable, or
- induced from a different smooth structure on $X_A(2) \cap \mathbb{R}^m$. 
Thank you very much for your attention.
Gam-sa-hap-ni-da.