Hamiltonian loops on symplectic blow ups

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The set up and the problems

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Problem.

- Understand the homotopy type of \(\text{Ham}(M, \omega)\).
- Determine \(\pi_1(\text{Ham}(M, \omega))\).
Known results

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- $\text{Ham}(S^2 \times S^2, \omega \oplus \lambda \omega)$ for $\lambda \geq 1$. (Gromov, McDuff, Abreu, Lalonde, etc.)
- $\text{Ham}(\mathbb{C}P^2, \omega_{FS}) \simeq PU(3)$. (Gromov)
Tools to compute $\pi_1$ and the goal of the talk

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- **Flux**: $\pi_1(\text{Symp}(M, \omega)) \to H^*(M; \mathbb{R})$
- **$S$**: $\pi_1(\text{Ham}(M, \omega)) \to QH^*(M, \Lambda)$
- **$A$**: $\pi_1(\text{Ham}(M, \omega)) \to \mathbb{R}/\mathcal{P}(M, \omega)$. 

Our aim is to show that some loops in $\text{Ham}(M, \omega)$ induced nontrivial loops in $\text{Ham}(\tilde{M}, \tilde{\omega}_\rho)$. $(\tilde{M}, \tilde{\omega}_\rho)$ is the one point blow up of weight $\rho$ of $(M, \omega)$. 


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$(\tilde{M}, \tilde{\omega}_\rho)$ : is the one point blow up of weight $\rho$ of $(M, \omega)$. 
Consider \((\mathbb{C}P^n, \omega_{FS})\) for \(n \geq 2\) and the Hamiltonian circle action

\[e^{2\pi it} \cdot [z_0 : \cdots : z_n] = [z_0 : e^{2\pi it} z_1 : \cdots : e^{2\pi int} z_n]\]

Let \(\psi\) be the corresponding Hamiltonian loop.
Example: Complex projective space

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- Blow up \((\mathbb{C}P^n, \omega_{FS})\) at \(x_0\) to get \((\tilde{\mathbb{C}}P^n, \tilde{\omega}_\rho)\)
- The loop \(\psi\) induces a Hamiltonian loop \(\tilde{\psi}\) on the blow up
Weinstein’s morphism

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In our example

- \( \mathcal{P}(\mathbb{C}P^n, \omega_{FS}) = \mathbb{Z}_\pi \)
- \( \mathcal{P}(\tilde{\mathbb{C}}P^n, \tilde{\omega}_\rho) = \mathbb{Z}_\pi + \mathbb{Z}_\pi \rho^2 \)
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and in the blow up, $\tilde{\psi} \in \text{Ham}(\hat{\mathbb{C}}P^n, \tilde{\omega}_\rho)$

$$A(\tilde{\psi}) = \left[ -\frac{n}{2}\pi + \frac{\pi \rho^{2n}}{2(1 - \rho^{2n})} \left( \frac{\rho^2}{(n-1)!} - n \right) \right] \in \mathbb{R}/\mathbb{Z}\langle \pi, \pi \rho^2 \rangle.$$
Results

Theorem (P.)

Let $\psi$ be the Hamiltonian loop defined above and $0 < \rho < 1$. Then $\psi$ induces a loop $\tilde{\psi}$ in $\text{Ham}(\widetilde{\mathbb{C}P^n}, \tilde{\omega}_\rho)$ and

- $\tilde{\psi}$ has finite order in $\pi_1(\text{Ham}(\mathbb{C}P^n, \omega))$ if $\rho^2$ is rational;
- $\tilde{\psi}$ has infinite order in $\pi_1(\text{Ham}(\mathbb{C}P^n, \omega))$ if $\rho^2$ is transcendental.
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Theorem (P.)

Let $(M, \omega)$ be a closed symplectic manifold and $(\tilde{M}, \tilde{\omega}_\rho)$ the blow up at $x_0 \in M$ of weight $\rho$. If $\psi \in \pi_1(\text{Ham}(M, \omega))$ has a representative that can be lifted to a loop $\tilde{\psi}$ in $\text{Ham}(\tilde{M}, \tilde{\omega}_\rho)$, then

$$\mathcal{A}_{\tilde{M}}(\tilde{\psi}) = \left[ \mathcal{A}_M(\psi) + \int_0^1 c_\rho(M, \omega, H_t) dt \right]$$

in $\mathbb{R}/\mathcal{P}(\tilde{M}, \tilde{\omega}_\rho)$ where $H_t$ is the normalized Hamiltonian function of the loop $\psi$. 