Combinatorial properties of subspace arrangements

Sangwook Kim

Chonnam National University

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Hyperplane arrangements

Simplicial complexes and subspace arrangements

Diagonal subspace arrangements

Coordinate subspace arrangements
1. Hyperplane arrangements

2. Simplicial complexes and subspace arrangements

3. Diagonal subspace arrangements

4. Coordinate subspace arrangements
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2. Simplicial complexes and subspace arrangements

3. Diagonal subspace arrangements

4. Coordinate subspace arrangements
Outline

1. Hyperplane arrangements
2. Simplicial complexes and subspace arrangements
3. Diagonal subspace arrangements
4. Coordinate subspace arrangements
Definition

A **subspace arrangement** is a finite collection of affine subspaces in the vector space $\mathbb{K}^n$ for some field $\mathbb{K}$.

Definition

A **hyperplane arrangement** is a subspace arrangement of codimension 1 subspaces.

There is a long tradition of work on hyperplane arrangements.

Definition

The intersection semilattice $L_\mathcal{A}$ of a subspace arrangement $\mathcal{A}$ is the collection of all nonempty intersections of subspaces of $\mathcal{A}$ ordered by reverse inclusion.
Subspace arrangements

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Combinatorial tradition ($\mathbb{R}$-arrangements)

**Theorem (Zaslavsky, 1975)**

Let $A$ be a hyperplane arrangement in $\mathbb{R}^n$. Then

The number of regions $= \sum_{x \in L_A} |\mu(\hat{0}, x)|$

The number of bounded regions $= |\sum_{x \in L_A} \mu(\hat{0}, x)|$

**Example**

12 regions
5 bounding regions
Theorem (Zaslavsky, 1975)

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- 12 regions
- 2 bounded regions
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Example

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Theorem (Orlik & Slomon, 1980)

Let $A$ be a hyperplane arrangement in $\mathbb{C}^d$ with complement $M_A$. Then

$$\beta^i(M_A) = \sum_{x \in L_A, \text{codim}_C(x) = i} |\mu(\hat{0}, x)|$$

where $\beta^i(M_A)$ is the rank of the cohomology group $H^i(M_A)$.

Two theorems are related

If $A$ is an $\mathbb{R}$-arrangement and $A^C$ is its complexification,

The number of regions $= \sum_{i \geq 0} \beta^i(M_{A^C})$

The number of bounded regions $= |\chi(M_{A^C})|$
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Two theorems are related

If $\mathcal{A}$ is an $\mathbb{R}$-arrangement and $\mathcal{A}^\mathbb{C}$ is its complexification,

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Outline

1. Hyperplane arrangements
2. Simplicial complexes and subspace arrangements
3. Diagonal subspace arrangements
4. Coordinate subspace arrangements
Simplicial complexes and shellability

Definition

An (abstract) simplicial complex $\Delta$ on a finite vertex set $V$ is a collection of subsets of $V$ satisfying

$$\tau \subset \sigma \in \Delta \Rightarrow \tau \in \Delta.$$ 

- $\dim \sigma = |\sigma| - 1$ and $\dim \Delta = \max_{\sigma \in \Delta} \dim \sigma$.
- The elements of $\Delta$ are faces and the maximal faces are facets.
- $\Delta$ is pure if each facet has the same dimension.

Definition

A simplicial complex is shellable if its facets can be arranged in linear order $F_1, F_2, \ldots, F_t$ in such a way that the subcomplex $(\bigcup_{i=1}^{k-1} 2^{F_i}) \cap 2^{F_k}$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, \ldots, t$. Such an ordering of facets is called a shelling order or shelling.
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Example

Theorem (Björner and Wachs, 1996)

A (nonpure) shellable simplicial complex is homotopy equivalent to

\[
\bigvee_{F \text{ runs over all fully attached facets}} S^\dim F
\]

where \(F\) runs over all fully attached facets.
Simplicial complexes and shellability

**Example**

![Diagram](image)

**Facets**

<table>
<thead>
<tr>
<th>$F$</th>
<th>$\mathcal{R}(F)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>234</td>
<td>4</td>
</tr>
<tr>
<td>35</td>
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Theorem (Björner and Wachs, 1996)

A (nonpure) shellable simplicial complex is homotopy equivalent to

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### Example

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#### Theorem (Björner and Wachs, 1996)

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Simplicial complexes and shellability

Example

Facets

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Facets

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\[ F_3 \]
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**Theorem (Björner and Wachs, 1996)**

A (nonpure) shellable simplicial complex is homotopy equivalent to

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The order complex of a poset

**Definition**

The order complex of a poset $P$ is the simplicial complex whose vertices are the elements of $P$ and whose faces are the chains of $P$.

**Example**

A poset $P$  

The order complex of $P$  

**Definition**

A finite lattice $L$ has some topological properties, such as shellability, if the order complex of $\overline{L} = L - \{\hat{0}, \hat{1}\}$ has those properties.
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A finite lattice $L$ has some topological properties, such as shellability, if the order complex of $\overline{L} = L - \{\hat{0}, \hat{1}\}$ has those properties.
Two important spaces associated with $\mathcal{A}$

**Definition**
- The complement of an arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is
\[ \mathcal{M}_\mathcal{A} = \mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H \]
- The singularity link of a central arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is
\[ \mathcal{V}^\circ_\mathcal{A} = S^{n-1} \cap \bigcup_{H \in \mathcal{A}} H \]

**Fact**
By Alexander duality,
\[ \tilde{H}^i(\mathcal{M}_\mathcal{A}; \mathbb{F}) = \tilde{H}_{n-2-i}(\mathcal{V}^\circ_\mathcal{A}; \mathbb{F}) \]
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What is the topology of $\mathcal{M}_A$ and $\mathcal{V}_A^\circ$?

Theorem (Goresky and Macpherson, 1988)

Let $A$ be a subspace arrangement in $\mathbb{R}^n$. Then

$$\tilde{\mathcal{H}}^i(\mathcal{M}_A) \cong \bigoplus_{x \in L_A - \{\hat{0}\}} \tilde{\mathcal{H}}_{\text{codim}(x) - 2 - i}(\hat{0}, x).$$

Theorem (Ziegler and Živaljević, 1993)

For every central subspace arrangement $A$ in $\mathbb{R}^n$,

$$\mathcal{V}_A^\circ \cong \bigvee_{x \in L_A - \{\hat{0}\}} (\Delta(\hat{0}, x) \ast S^{\dim(x) - 1}).$$
What is the topology of $M_{\mathcal{A}}$ and $V_{\mathcal{A}}^\circ$?

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### Correspondence

A simplicial complex \( \Delta \) on \([n]\) \(\iff\) A diagonal arrangement \( A_\Delta \):

- Collection of diagonal subspaces \( \{x_{i_1} = \cdots = x_{i_k}\} \) of \( \mathbb{R}^n \)
- For all \( \{i_1, \ldots, i_k\} \) complementary to facets of \( \Delta \)

### Example

- \( \Delta \)
- \( A_\Delta \)
### Correspondence

A simplicial complex $\Delta$ on $[n]$ isomorphic to $\Delta$ via

$$\begin{align*}
\Delta & \iff \\
\{x_{i_1} = \cdots = x_{i_k}\} & \text{diagonal subspaces of } \mathbb{R}^n, \\
\text{for all } \{i_1, \ldots, i_k\} & \text{complementary to facets of } \Delta
\end{align*}$$

### Example

\[ \Delta \]
A simplicial complex \( \Delta \) on \([n]\) \iff\ A diagonal arrangement \( \mathcal{A}_\Delta \): collection of diagonal subspaces \( \{x_{i_1} = \cdots = x_{i_k}\} \) of \( \mathbb{R}^n \) for all \( \{i_1, \ldots, i_k\} \) complementary to facets of \( \Delta \)

Example

\( \Delta \)

\( F_1 \quad F_2 \quad F_3 \quad F_4 \)

\( A_\Delta \)
Simplicial complexes and diagonal arrangements

**Correspondence**

A simplicial complex $\Delta$ on $[n] \iff$

A diagonal arrangement $\mathcal{A}_\Delta$: collection of diagonal subspaces

$$\{x_{i_1} = \cdots = x_{i_k}\} \text{ of } \mathbb{R}^n$$

for all $\{i_1, \ldots, i_k\}$ complementary to facets of $\Delta$

**Example**

$$\Delta$$

$\iff$

$$\{x_4 = x_5\}$$

$$\{x_1 = x_5\}$$

$$\{x_1 = x_2 = x_4\}$$

$$\{x_1 = x_2 = x_3\}$$
A simplicial complex $\Delta$ on $[n]$ is \(\iff\) a diagonal arrangement $\mathcal{A}_\Delta$: collection of diagonal subspaces $\{x_{i_1} = \cdots = x_{i_k}\}$ of $\mathbb{R}^n$ for all $\{i_1, \ldots, i_k\}$ complementary to facets of $\Delta$.

**Example**

$$\Delta$$

\[\begin{align*}
\{x_4 = x_5\} & \iff \{x_1 = x_5\} \\
\{x_1 = x_2 = x_4\} & \iff \{x_1 = x_2 = x_3\}
\end{align*}\]
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Example

\( \Delta \) \iff\ \{x_4 = x_5\} \quad \{x_1 = x_5\} \quad \{x_1 = x_2 = x_4\} \quad \{x_1 = x_2 = x_3\} \\
\mathcal{A}_\Delta \quad F_1 \quad F_2 \quad F_3 \quad F_4
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Example

\( \Delta \)

\( \mathcal{A}_\Delta \)

\( \{x_4 = x_5\} \)
\( \{x_1 = x_5\} \)
\( \{x_1 = x_2 = x_4\} \)
\( \{x_1 = x_2 = x_3\} \)
Example

The **Braid arrangement** $\mathcal{B}_n = \bigcup_{i<j} \{x_i = x_j\}$

$\uparrow$

$\Delta_{n,n-2} = \{\sigma \subset [n] : |\sigma| \leq n - 2\}$

Example

The **k-equal arrangement** $\mathcal{A}_{n,k} = \bigcup_{i_1 < \cdots < i_k} \{x_{i_1} = \cdots = x_{i_k}\}$

$\uparrow$

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Example

The Braid arrangement \( B_n = \bigcup_{i<j} \{ x_i = x_j \} \)

\[ \Delta_{n,n-2} = \{ \sigma \subset [n] : |\sigma| \leq n - 2 \} \]

Example

The \( k \)-equal arrangement \( A_{n,k} = \bigcup_{i < \cdots < i_k} \{ x_{i_1} = \cdots = x_{i_k} \} \)

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Example

The Braid arrangement \( \mathcal{B}_n = \bigcup_{i<j} \{x_i = x_j\} \)

\[ \Delta_{n,n-2} = \{\sigma \subseteq [n] : |\sigma| \leq n-2\} \]

Example

The \( k \)-equal arrangement \( \mathcal{A}_{n,k} = \bigcup_{i_1<\ldots<i_k} \{x_{i_1} = \ldots = x_{i_k}\} \)

\[ \Delta_{n,n-k} = \{\sigma \subseteq [n] : |\sigma| \leq n-k\} \]
Example

The Braid arrangement $\mathcal{B}_n = \bigcup_{i<j} \{x_i = x_j\}$

$\Delta_{n,n-2} = \{\sigma \subset [n] : |\sigma| \leq n - 2\}$

Example

The $k$-equal arrangement $\mathcal{A}_{n,k} = \bigcup_{i_1<\cdots<i_k} \{x_{i_1} = \cdots = x_{i_k}\}$

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What is a general sufficient condition for the intersection lattice $L_{\mathcal{A}}$ of a diagonal arrangement $\mathcal{A}$ to be well-behaved?

**Theorem (Björner and Welker, 1995)**

The order complex of the intersection lattice $L_{\mathcal{A}_{n,k}}$ for the $k$-equal arrangement $\mathcal{A}_{n,k}$ is shellable.

$\mathcal{A}_{n,k} = \mathcal{A}_{\Delta_{n,n-k}}$ and $\Delta_{n,n-k}$ is shellable.

**Theorem (Kozlov, 1999)**

Let $\Delta$ be a simplicial complex on $[n]$ that satisfies some conditions. Then the intersection lattice for $\mathcal{A}_{\Delta}$ is shellable.

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Homotopy type of the singularity link

Theorem (K.)

Let $\Delta$ be a shellable simplicial complex with $\dim \Delta \leq n - 3$. Then the order complex of the intersection lattice $L_\Delta$ of $A_\Delta$ is homotopy equivalent to a wedge of spheres.

Corollary (K.)

Let $\Delta$ be a shellable simplicial complex with $\dim \Delta \leq n - 3$. The singularity link of $A_\Delta$ has the homotopy type of a wedge of spheres.
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A shellable complex $\Delta$

The intersection lattice $L_\Delta$ of $\mathcal{A}_\Delta$
Example

A shellable complex $\Delta$

The intersection lattice $L_\Delta$ of $A_\Delta$
Example

A shellable complex \( \Delta \)

The intersection lattice \( L_\Delta \) of \( \mathcal{A}_\Delta \)

The order complex of \( \bar{L}_\Delta \)

\[ \{(12345, F_4)\} \quad \{(45, F_1), (123, F_4)\} \]
Example

A shellable complex $\Delta$

$\{(12345, F_4)\}$
$\{(45, F_1), (123, F_4)\}$

Shelling-trapped decompositions of $[5]$

The intersection lattice $L_\Delta$ of $\mathcal{A}_\Delta$

The order complex of $\overline{L_\Delta}$
Example

A shellable complex $\Delta$

\[
\{(12345, F_4)\} \cup \{(45, F_1), (123, F_4)\}
\]

Shelling-trapped decompositions of $[5]$

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Example

A shellable complex $\Delta$

\[
\{(12345, F_4)\}
\]
\[
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\]

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The order complex of $\overline{L_\Delta}$
$L_\Delta$ is not shellable in general

**Example**

Let $\Delta$ be a shellable complex with a shelling $123456, 127, 137, 237, 458, 468, 568$. 
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Let $\Delta$ be a shellable complex with a shelling 123456, 127, 137, 237, 458, 468, 568.
$\triangle$ is not shellable in general

Example

Let $\triangle$ be a shellable complex with a shelling $123456, 127, 137, 237, 458, 468, 568$. 

\[
\begin{array}{c}
12345678 \\
12345678 \\
12345678 \\
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12345678 \\
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12345678 \\
12345678 \\
12345678 \\
12345678 \\
12345678 \end{array}
\]
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Application in group cohomology

**Definition**

An Eilenberg-MacLane space (or a $K(\pi, n)$ space) is a connected cell complex with all homotopy groups except the $n$-th homotopy group being trivial and the $n$-th homotopy group isomorphic to $\pi$.

**Fact**

If a CW complex $X$ is a $K(\pi, 1)$ space, then

$$\text{Tor}_{n}^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = H_{n}(X; \mathbb{Z}) \text{ and } \text{Ext}_{\mathbb{Z}}^{n}(\mathbb{Z}, \mathbb{Z}) = H^{n}(X; \mathbb{Z}).$$

**Theorem (Fadell and Neuwirth, 1962)**

Let $B_{n}$ be the braid arrangement in $\mathbb{C}^{n}$. Then $M_{B_{n}}$ is a $K(\pi, 1)$ space.

**Theorem (Khovanov, 1996)**

Let $A_{n,3}$ be the 3-equal arrangement in $\mathbb{R}^{n}$. Then $M_{A_{n,3}}$ is a $K(\pi, 1)$ space.
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Diagonal arrangement $\mathcal{A}$ such that $\mathcal{M}_\mathcal{A}$ is $K(\pi, 1)$

**Theorem (Davis, Januszkiewicz and Scott, 1998)**

Let $\mathcal{H}$ be a simplicial real hyperplane arrangement in $\mathbb{R}^n$. Let $\mathcal{A}$ be any arrangement of codimension-2 intersection subspaces in $\mathcal{H}$ which intersects every chamber in a codimension-2 subcomplex. Then $\mathcal{M}_\mathcal{A}$ is $K(\pi, 1)$.

**Proposition**

Let $\mathcal{A}$ be a subarrangement of 3-equal arrangement of $\mathbb{R}^n$ so that

$$\mathcal{A} = \left\{ \{x_i = x_j = x_k\} \mid \{i, j, k\} \in T_\mathcal{A} \right\},$$

for some collection $T_\mathcal{A}$ of 3-element subsets of $[n]$. Then $\mathcal{A}$ satisfies the hypothesis of DJS’s theorem (and hence $\mathcal{M}_\mathcal{A}$ is $K(\pi, 1)$) if and only if every permutation $\omega$ in $\mathfrak{S}_n$ has at least one triple in $T_\mathcal{A}$ consecutive.
Diagonal arrangement $A$ such that $M_A$ is $K(\pi, 1)$

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DJS matroids

The matroid complexes $\Delta = \mathcal{I}(M)$ are a natural class of shellable complexes.

**Definition**

Say a rank 3 matroid $M$ on $[n]$ is **DJS** if every permutation $\omega$ in $\mathfrak{S}_n$ has at least one triple in $\mathcal{B}(M)$ consecutive.

**Proposition (K.)**

Rank 3 Matroids without parallel elements are DJS. In particular, rank 3 simple matroids are DJS.

**Proposition (K.)**

Let $M$ be a rank 3 matroid on the ground set $[n]$ with no circuits of size 3. Let $P_1, \ldots, P_k$ be distinct parallel classes which have more than one element and let $N$ be the set of all elements which are not parallel with anything else. Then, $M$ is DJS if and only if
\[
\left\lfloor \frac{|P_1|}{2} \right\rfloor + \cdots + \left\lfloor \frac{|P_k|}{2} \right\rfloor - k < |N| - 2.
\]
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1. Hyperplane arrangements
2. Simplicial complexes and subspace arrangements
3. Diagonal subspace arrangements
4. Coordinate subspace arrangements
### Correspondence

A simplicial complex $\Delta$ on $[n] \iff$ A coordinate arrangement $\mathcal{A}_\Delta$:

- collection of coordinate subspaces
- $\{x_{i_1} = \cdots = x_{i_k} = 0\}$ of $\mathbb{R}^n$
- for all $\{i_1, \ldots, i_k\}$ complementary to facets of $\Delta$

### Example

- $F_1$
- $F_2$
- $F_3$
- $F_4$
Simplicial complexes and Coordinate arrangements

Correspondence

A simplicial complex \( \Delta \) on \([n]\) \iff

A coordinate arrangement \( A_\Delta \): collection of coordinate subspaces \( \{x_{i_1} = \cdots = x_{i_k} = 0\} \) of \( \mathbb{R}^n \) for all \( \{i_1, \ldots, i_k\} \) complementary to facets of \( \Delta \)

Example
### Correspondence

A simplicial complex $\Delta$ on $[n]$ \iff A coordinate arrangement $A_\Delta$: collection of coordinate subspaces $\{x_{i_1} = \cdots = x_{i_k} = 0\}$ of $\mathbb{R}^n$ for all $\{i_1, \ldots, i_k\}$ complementary to facets of $\Delta$

### Example

- $\Delta$
- $A_\Delta$
Correspondence

A simplicial complex $\Delta$ on $[n] \iff$ A coordinate arrangement $\mathcal{A}_\Delta$: collection of coordinate subspaces

\[
\{x_{i_1} = \cdots = x_{i_k} = 0\}
\]

of $\mathbb{R}^n$ for all \(\{i_1, \ldots, i_k\}\) complementary to facets of $\Delta$

Example

\[
\begin{align*}
\{x_4 = x_5 = 0\} & \quad \Rightarrow \quad F_1 \\
\{x_1 = x_6 = 0\} & \quad \Rightarrow \quad F_2 \\
\{x_1 = x_2 = x_4 = 0\} & \quad \Rightarrow \quad F_3 \\
\{x_1 = x_2 = x_3 = 0\} & \quad \Rightarrow \quad F_4
\end{align*}
\]
A simplicial complex \( \Delta \) on \([n]\) \iff \text{A coordinate arrangement} \( \mathcal{A}_\Delta \) :

- collection of \text{coordinate} subspaces \( \{x_{i_1} = \cdots = x_{i_k} = 0\} \) of \( \mathbb{R}^n \)
- for all \( \{i_1, \ldots, i_k\} \) \text{complementary to facets of} \( \Delta \)

\[\begin{align*}
\{x_4 = x_5 = 0\} & \iff F_1 \\
\{x_1 = x_5 = 0\} & \iff F_2 \\
\{x_1 = x_2 = x_4 = 0\} & \iff F_3 \\
\{x_1 = x_2 = x_3 = 0\} & \iff F_4
\end{align*}\]
Lemma

Let $\Delta$ be a simplicial complex and $L_{\Delta}$ be the intersection lattice for its corresponding coordinate arrangement. Then the subspace $x_{i_1} = \cdots = x_{i_k} = 0$ lies in $L_{\Delta}$ if and only if $[n] - \{i_1, \ldots, i_k\}$ is an intersection of facets of $\Delta$.

Proposition

Let $\Delta$ be a simplicial complex on $[n]$ and $\sigma$ be the intersection of all facets of $\Delta$. Then the intersection lattice $L_{\Delta}$ is homotopy equivalent to $\text{link}_{\Delta} \sigma$. 
Lemma

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A simplicial complex $\Delta$

Intersection lattice $L_\Delta$

Barycentric subdivision of $\Delta$

The order complex of $\overline{L_\Delta}$
A simplicial complex $\Delta$

Intersection lattice $L_\Delta$

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The order complex of $\overline{L_\Delta}$
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A simplicial complex $\Delta$

Intersection lattice $L_\Delta$

Barycentric subdivision of $\Delta$

The order complex of $\overline{L_\Delta}$
Example

A simplicial complex $\Delta$

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Intersection lattice $L_{\Delta}$

The order complex of $\overline{L_{\Delta}}$
A simplicial complex $\Delta$

Barycentric subdivision of $\Delta$

Intersection lattice $L_\Delta$

The order complex of $\overline{L_\Delta}$
**Proposition**

If $\Delta$ is a shellable simplicial complex, then the singularity link of $A_\Delta$ is homotopy equivalent to

$$\bigvee_{i=1}^{q} \left( \bigvee_{\dim F_i \leq \dim F_i} \bigvee_{2|\mathcal{R}(F_i)| \text{ copies}} S_{\dim F_i} \right),$$

where $\mathcal{R}(F_i)$ is the unique minimal new face of $F_i$.

**Conjecture (Welker)**

If $\Delta$ is a shellable simplicial complex, then the complement of $A_\Delta$ is homotopy equivalent to

$$\bigvee_{i=1}^{q} \left( \bigvee_{n-2-\dim F_i \leq \dim F_i} \bigvee_{2|\mathcal{R}(F_i)| \text{ copies}} S_{n-2-\dim F_i} \right).$$
Proposition

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Proposition

If a simplicial complex \( \Delta \) is shellable, then the multiplication on the cohomology algebra of the complement of \( A_\Delta \) is trivial.

Sketch of proof

- The cohomology algebra of the complement of \( A_\Delta \) is isomorphic to Tor algebra of the Stanley-Reisner ring \( k[\Delta] \).
- If a simplicial complex \( \Delta \) is shellable, then the corresponding Stanley-Reisner ring \( k[\Delta] \) is Golod.
- A monomial ring is Golod if and only if the multiplication on its Tor algebra is trivial. [Berglund and Jöllenbeck, 2007]

Note

If the complement of \( A_\Delta \) is homotopy equivalent to a wedge of spheres, then the multiplication on its cohomology algebra is trivial.
Proposition

If a simplicial complex $\Delta$ is shellable, then the multiplication on the cohomology algebra of the complement of $A_\Delta$ is trivial.

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If the complement of $A_\Delta$ is homotopy equivalent to a wedge of spheres, then the multiplication on its cohomology algebra is trivial.
Proposition

If a simplicial complex $\Delta$ is shellable, then the multiplication on the cohomology algebra of the complement of $\mathcal{A}_\Delta$ is trivial.

Sketch of proof

- The cohomology algebra of the complement of $\mathcal{A}_\Delta$ is isomorphic to Tor algebra of the Stanley-Reisner ring $k[\Delta]$.
- If a simplicial complex $\Delta$ is shellable, then the corresponding Stanley-Reisner ring $k[\Delta]$ is Golod.
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Cohomology algebra of the complement of \( A_\Delta \)

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If the complement of \( A_\Delta \) is homotopy equivalent to a wedge of spheres, then the multiplication on its cohomology algebra is trivial.
A simplicial complex $\Delta$ on $[n]$ is **shifted** if, for any face of $\Delta$, replacing any vertex $i$ by a vertex $j(< i)$ gives another face in $\Delta$.

**Theorem (Björner and Wachs, 1996)**

Shifted complexes are shellable.

**Theorem (Grbić & Theriault, 2007 / Welker)**

If $\Delta$ is a shifted simplicial complex, then the complement of $A_\Delta$ is homotopy equivalent to a wedge of spheres.
Known cases - shifted complexes

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Proposition (K.)

If $\Delta$ is a pure and shellable simplicial complex on $[n]$ with $d := \dim \Delta \leq n - 3$, then the complement of $\mathcal{A}_\Delta$ is homotopy equivalent to

$$S^{n-d-2} \vee \ldots \vee S^{n-d-2} \quad \sum_{i=1}^{q} 2^{\left|\mathcal{R}(F_i)\right|} \text{ copies}.$$

Sketch of proof

- $\Gamma_\Delta :=$ the faces of $n$-cube $C^n = [-1, 1]^n$ disjoint to $\cup \mathcal{A}_\Delta$
- $\Gamma_\Delta \simeq \mathcal{M}_{\mathcal{A}_\Delta}$
- $H_i(\mathcal{M}_{\mathcal{A}_\Delta}; \mathbb{Z}) = \begin{cases} \sum_{i=1}^{q} 2^{\left|\mathcal{R}(F_i)\right|} & i = n - d - 2 \\ 0 & \text{otherwise} \end{cases}$
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Sketch of proof

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Known cases - pure and shellable complexes

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When a complex is not shellable

Example

Nonshellable complex $\Delta$  Coordinate arrangement $\mathcal{A}_\Delta$

- The complement of $\mathcal{A}_\Delta$ is

$$\mathbb{R}^4 - \{x_1 = x_2 = 0\} \cup \{x_3 = x_4 = 0\}$$

which is homotopy equivalent to a torus.

- The singularity link of $\mathcal{A}_\Delta$ is homotopy equivalent to a wedge of spheres.
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Coordinate arrangement $\mathcal{A}_\Delta$
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Nonshellable complex $\Delta$

- The complement of $\mathcal{A}_\Delta$ is
  \[ \mathbb{R}^4 - \left[ \{ x_1 = x_2 = 0 \} \cup \{ x_3 = x_4 = 0 \} \right] \]
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Coordinate arrangement $\mathcal{A}_\Delta$

\[ x_1 = x_2 = 0 \]
\[ x_3 = x_4 = 0 \]
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Example

Nonshellable complex $\Delta$

Coordinate arrangement $A_\Delta$

- The complement of $A_\Delta$ is

$$\mathbb{R}^4 \setminus \{x_1 = x_2 = 0\} \cup \{x_3 = x_4 = 0\}$$

which is homotopy equivalent to a torus.

- The singularity link of $A_\Delta$ is homotopy equivalent to a wedge of spheres.

$$x_1 = x_2 = 0$$

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When a complex is not shellable

Example

Nonshellable complex $\Delta$

Coordinate arrangement $\mathcal{A}_\Delta$

- The complement of $\mathcal{A}_\Delta$ is

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- The singularity link of $\mathcal{A}_\Delta$ is homotopy equivalent to a wedge of spheres.
Thank you for your attention!!