Homomorphic Encryptions and Private Set Operations

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Contents

- Privacy Homomorphism
- Asymetric Homomorphic Encryption
- Set Operations on Encrypted Data
- Practical Applications

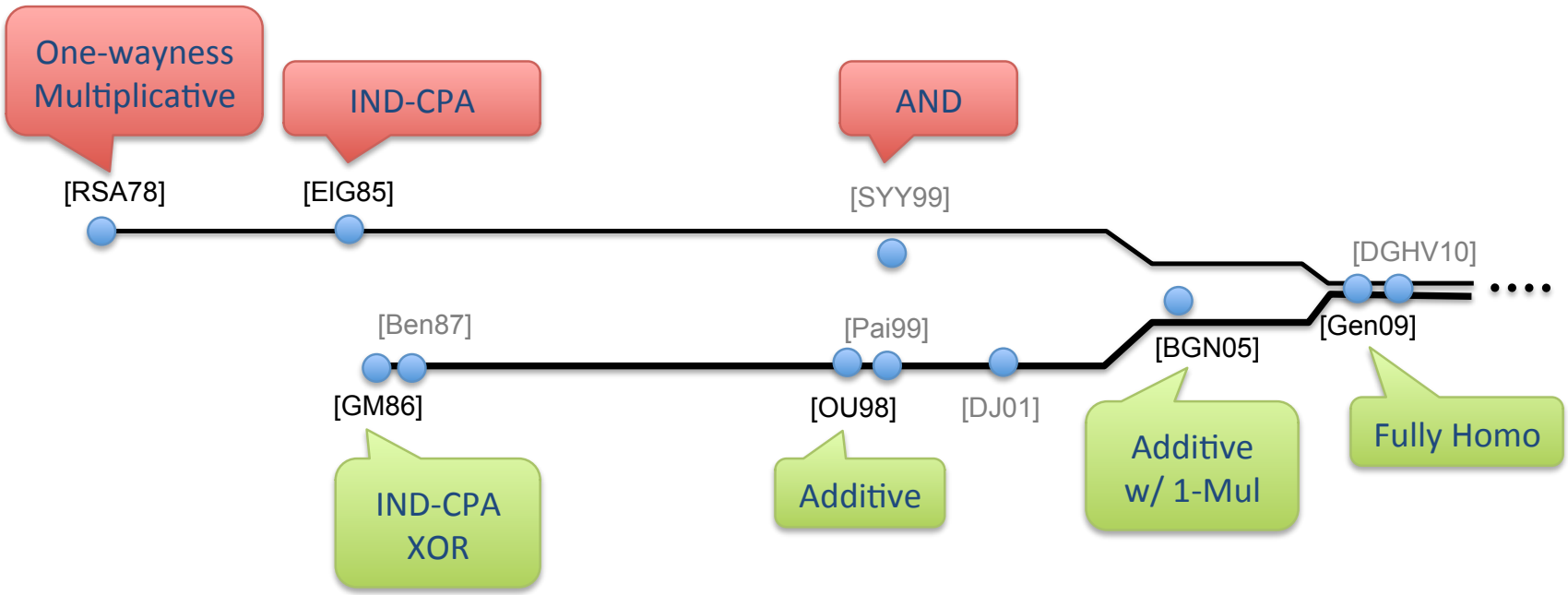
What We Need

- Computing on Encrypted Data
- Homomorphic Encryptions ...

Fully Homomorphic Encryption

- Overkill or not?
- Is there either jump or overlook?

Brief History



Homomorphic Schemes

- Privacy Homomorphism: [RAD78]
- XOR: [GM86]
- AND: [SYY99]
- Addition: [Ben87], [OU98], [NS98],[Pai99], [DJ01] and so on...
- Multiplication: [RSA], [EIG85]
- Addition + d-Multiplication: [BGN05],[MGH10]
- Full Operations: [Gen09], [DGHV10], and so on...

Privacy Homomorphism

- "encryption functions which permit encrypted data to be operated on without preliminary decryption of the operands, for many sets of interesting operations"
- Data can be processed by an external computing facility (Database as a service)
- Suppose the loan company stores their data in a outsourced database [RAD78]
  - What is the size of the average loan outstanding?
  - How much income from loan payments is expected next month?
  - How many loans over $\$5,000$ have been granted?

PH by Rivest et al.

- Private key: two large primes p and q
- Public key: $n = pq$
- Encryption: $E_n(a) = (a \bmod p, a \bmod q)$ for $a \in \mathbb{Z}_n$
- Decryption:
  $D_{p,q}(d_1, d_2) = d_1 q(q^{-1} \bmod p) + d_2 p(p^{-1} \bmod q) \bmod n$
- Support modular addition, subtraction, and multiplication
- Very efficient!

Security of RAD-PH

- Secure under no message attack
- What if we have a pair of pt and ct? (Brickell and Yacobi)
- Introducing random factors...
  - Take $N = \prod p_i$ and $E : Z_N \rightarrow Z_N$.
  - $E(m) = c$, where $c \equiv (m + e_i) \bmod p_i$ for small e_i.
- A homomorphic encryption is insecure under the chosen ciphertext attack

Research Direction

- Construct an efficient additive homomorphic *symmetric* encryption
- Construct a secure ring homomorphic *symmetric* encryption

XOR Homomorphic Encryption

Goldwasser-Micali Scheme [GM86]

- Basic Idea: Map 0 to a random QR and 1 to a random QNR
- on the QR assumption & IND-CPA security
- Work flow
  - KeyGen(1^λ): $pk = (n = pq, \alpha)$ and $sk = (p, q)$
    – $\alpha \notin \mathrm{QR}_n$ s.t $\left(\frac{\alpha}{p}\right) = -1$ and $\left(\frac{\alpha}{q}\right) = -1$.
  - Enc(pk, m): To encrypt a message $m = \{0, 1\}$
    – $c = \alpha^m r^2 \pmod n$ for a random $r \xleftarrow{\$} \mathbb{Z}_n^\times$
  - Dec(sk, c):
    – $m = 0$ if $c \in \mathrm{QR}$; otherwise $m = 1$
- Homomorphism
  – $c_1 c_2 = \alpha^{m_1} r_1^2 \cdot \alpha^{m_2} r_2^2 = \alpha^{(m_1 + m_2 \mod 2)} (r_1 r_2)^2$

AND Homomorphic Encryptions

The Sander-Young-Yung Encryption [SYY99]

- Basic Idea: Special Encoding/Decoding + [GM86]
  - Encode $\{0,1\} \to \{0,1\}^\ell$, $0 \mapsto r \in_R \{0,1\}^\ell$ and $1 \mapsto 0^\ell$
  - Encrypt each bit in the ℓ-bit vector using HE $\mathcal{E}$ of [GM86]
  - Decrypt and decode (msg is 1 iff output is a zero vector)
- Homomorphism
  - $\mathrm{Enc}(m_1) = (\mathcal{E}(a_1), \ldots \mathcal{E}(a_\ell))$ and $\mathrm{Enc}(m_2) = (\mathcal{E}(b_1), \ldots \mathcal{E}(b_\ell))$ where $a_i, b_i \in \{0,1\}$
  - $\mathrm{Enc}(m_1) \wedge \mathrm{Enc}(m_2) = (\mathcal{E}(a_1) \cdot \mathcal{E}(b_1), \ldots, \mathcal{E}(a_\ell) \cdot \mathcal{E}(b_\ell))$
  - Note random $\oplus$ random $=$ random, random $\oplus\, 0^\ell =$ random, $0^\ell \oplus 0^\ell = 0^\ell$
- Message expansion: ℓn

Research Direction

- Construct an homomorphic encryption w.r.t an operation which is functionally complete
- NOR, NAND, ...
- Asymmetric or symmetric

Homomorphic in mod M multiplication

- RSA encryption
  - on the RSA assumption
  - One-wayness security
  - M is a secret composite number (i.e. $\phi(N)$)
- ElGamal encryption
  - on the DDH assumption
  - IND-CPA security
  - M is a public prime
  - Variants based on different hardness assumptions

Additively Homomorphic in $\mathbb{Z}_N$

The Benaloh Encryption [Ben87]

- on the ℓ-th Residuosity Problem
- Basic idea: Remove random part by unknown order and find exhaustively a correct message in $\mathbb{Z}_\ell$
- Message expansion $= \frac{n}{\ell}$ (vs [GM86]: n)

The Naccache-Stern Encryption [NS98]

- on the Factoring n Problem
- Basic idea: Messages are represented by small primes and reconstructed using CRT based on [Ben87]
- Message expansion ≥ 4

Example of NS98

- Outline
  - Message space $\mathcal{M} = \mathbb{Z}_M$ where $M = 2^a 3^b$
  - Decryption utilizes Pohlig-Hellman for solving DL
  - In [OU98] M is unknown prime and in [Pai99] $M = n$ is a hard-to-factor composite
- Scheme
  - KeyGen: $pk = (n = pq, g, h, a, b)$ and $sk = (p, q)$ s.t $p - 1 = 2^a p', q - 1 = 3^b q'$ for some primes p', q', and g is a random generator of order $\lambda(n) = Mp'q'$ and h a random element of order $p'q'$
  - Enc(pk, m): To encrypt $m \in \mathbb{Z}_M$
    – $c = g^m h^r \mod n$ for some random $r \in [1, p'q']$
  - Dec(sk, c):
    – Compute $c^{p'q'} = (g^{p'q'})^m$ and solve DL using Pohlig-Hellman

The Okamoto-Uchiyama Encryption [OU98]

- on the Factoring $n = p^2 q$ Problem/Sylow p-subgroup Problem
- Basic idea:
  - H: the unique subgroup of order p of $\mathbb{Z}_{p^2}^{\times} \subset \mathbb{Z}_n^{\times}$
  - Solve the DLP on $H \subset \mathbb{Z}_n^{\times}$ easily.
- Logarithm
  - $(1+p)^x \equiv 1 + xp \bmod p^2$ and $(1+p)^p \equiv 1 \bmod p^2$
  - $\zeta := 1 + p$ is an order-p element of $\mathbb{Z}_{p^2}^*$.
  - Given $h \in H$, exist $x \in \mathbb{Z}_p$ s.t. $h = \zeta^x$. Then $x := (h-1)/p \bmod p$.
  - Define an isom $L = \log_\zeta : H \to \mathbb{Z}_p$, $h \mapsto (h-1)/p$
  - Given $g, h \in H$, $\log_g h = (\log_\zeta h)/(\log_\zeta g) = L(h)/L(g) = (h-1)/(g-1)$

The Okamoto-Uchiyama Encryption [OU98]

- Work flow
  - KeyGen: $pk = (n = p^2q, g, h, \ell)$ and $sk = (p, q)$ where p, q: ℓ-bit primes, $g \in_R \mathbb{Z}_n^\times$ s.t $g^{\varphi(p^2)} = 1 \mod p^2$ and $g^{p-1} \neq \mod p^2$, and $h = g^n \mod n$
  - Enc(pk, m): to encrypt $m \in \mathbb{Z}_p$
    – $c = g^m h^r \mod n$ for some random $r \in \mathbb{Z}_n$
  - Dec(sk, c)
    – $m = \frac{L(c^{p-1} \mod p^2)}{L(g^{p-1} \mod p^2)} \mod p$
- Message expansion $= 3$

The Paillier Encryption [Pai99]

- Drawbacks of [OU98]: Decryption oracle enables factoring N
- Consider the modulus N^2 for $N = pq$ instead of p^2q. Then the message is defined over Z_N
- Logarithm
  - $(1+N)^x \equiv 1 + xN \bmod N^2$ and $(1+N)^N \equiv 1 \bmod N^2$
  - $\zeta := 1 + N$ is an order-N element of $\mathbb{Z}^*_{N^2}$.
  - Given $h \in H$, we have $h = \zeta^x$ for $x := (h-1)/N \bmod N$.
  - Define an isom $L : H \to \mathbb{Z}_N$, $h \mapsto (h-1)/N$
  - Given $g, h \in H$, $\log_g h = (\log_\zeta h)/(\log_\zeta g) = (h-1)/(g-1)$

The Paillier Encryption [Pai99]

- $\mathcal{E}(m) = g^m h^r \bmod N^2$ for $g = g_0^{\Phi(N)}$ and $h = g_0^N$.
- $\mathcal{E}(m)^{\phi(N)} \equiv g^m \bmod N^2$. Then $L(g^m)/L(g) \equiv m \bmod N$
- Paillier: $\mathcal{E}(m) = g^m r^N \bmod N^2$ for $g = 1 + N$ and random $r \in \mathbb{Z}_{N^2}^*$
- Use $\lambda(N)$ instead of $\phi(N)$
- Variants: EC version [Gal02], Generalized version [DJ01]

Additively Homomorphic Encryption with 2-Mul.

The Boneh-Goh-Nissim Encryption [BGN05]

- on the Subgroup Decision Problem
- Basic idea
  - $|\mathbb{G}| = pg$, $|g| = p$ and $|h| = q$. $\mathcal{E}(m) = g^m h^r$ is add homo on $\mathbb{Z}_p$
  - Use a bilinear map $e : \mathbb{G} \times \mathbb{G} \to \mathbb{G}_1$
- Allow one mul: $e(g^{m_1} h^{r_1}, g^{m_2} h^{r_2}) = e(g^{m_1} g^{\alpha r_1}, g^{m_2} g^{\alpha r_2})$ $= e(g, g)^{(m_1+\alpha r_1)(m_2+\alpha r_2)} = g_1^{m_1 m_2}$. ♣
- How to decrypt ..
- Scalar Mul is not possible is p is large

Fully Homomorphic Encryption

The Gentry Encryption [Gen09]

- Hard Problems on Ideal Lattice
- Basic idea
  - Build somewhat homomorphic encryption whose ciphertexts becomes impossible to decrypt after bounded homomorphic operations
  - Homomorphic evaluation of its own decryption circuit $\Rightarrow$ Flush accumulated noise
    – Imagine a locked box containing another locked box, but in the outer box has a key to open the inner box $\Rightarrow$ in the outer box, we can safely disclose the inner box
- Variants: [DGHV10] and so on

Fully Homomorphic Encryption [DGHV10]

- Work flow (Symmetric version of [DGHV10])
  - KeyGen(1^λ): the key is λ^2-bit odd integer p
  - Enc(p, m): To encrypt $m \in \{0, 1\}$
    – $c = m' + pq$, where m' is a λ-bit number s.t $m' = m \mod 2$ and q a λ^5-bit random number
  - Dec(p, c)
    – $m = (c \mod p) \mod 2$
- How homomorphic is it?
- Efficiency (in [DGHV10])
  - Ciphertext size: $\lambda^5 \cdot (\log \lambda)^k$ bits for some k
  - Decryption function evaluation: $\lambda^{10} \cdot (\log \lambda)^k$ computation
- Security: Approximate GCD problem

Additively Homomorphic Encryption with d-Mul.

The Melchor-Gaborit-Herranz Encryption [MGH10]

- on the Unique Shortest Vector Problem
- Basic idea
  - Framework to convert an additive homomorphic encryption based on lattice problem into an encryption supporting arbitrary addition with 2-multiplication

Research Direction

- BGN with large plaintext
- Additive HE allowing d multiplications
- Efficient Somewhat Homomorphic Enc using Lattice
- Optimited construction to specific applications
- ...

Ciphertext Operations

1. Numeric Data: Additive Homomorphic Encryption, Multiplicative Homomorphic Encryption
   - Basic Integer Operations: Add, Subtract, Mul, Div
   - Advanced Operations: Euclidean Alg, GCD, Modular Operations
   - Fast Operations: Gaussian Elimination, Newton Method, FFT
2. Non-Numeric Data: what is the operation of data?
   - Search
   - Intersection, Union and Difference

Encrypted Set Operations

1. We wish to design an encryption scheme satisfying ...
   - Intersection: $\mathcal{E}(A \cup B)$ from $\mathcal{E}(A), E(B)$
   - Union: $\mathcal{E}(A \cap B)$ from $\mathcal{E}(A), \mathcal{E}(B)$
   - Difference: $\mathcal{E}(A \setminus B)$ from $\mathcal{E}(A), \mathcal{E}(B)$

   - Find $\mathcal{E}(A \cup B)$ or $\mathcal{E}(A \cap B)$ from A and $\mathcal{E}(B)$
     - Reduced to the above for PKE $\mathcal{E}$
   - Find $A \cup B$ or $A \cap B$ from A and $\mathcal{E}(B)$
     - Keyword Privacy: Use encrypted keywords
2. Easy Solution: Use deterministic encryption for individual data :-)

Polynomial Representation of Sets

1. S_i: a subset of $\mathbb{Z}_N$
2. Poly Rep of S_i: $f_i(x) = (x - \alpha_1) \cdots (x - \alpha_k)$ for $a_j \in S_i$
   - $V(f)$: the set of all roots of $f(x)$ (in $\mathbb{Z}_N$)
   - $V(f_i) = S_i$
3. Multiset Union: $V(f_1(x) \cdot f_2(x)) = S_1 \cup S_2$
4. (Multi)Set Intersection: $V(af_1(x) + bf_2(x)) \supset S_1 \cap S_2$
   - If a, b are polynomials of deg f_1=deg f_2, $af_1 + bf_2 = c \gcd(f_1, f_2)$ for a poly c uniformly distributed over a fixed deg poly.

Set Encryption

1. E: an Additive Homo Enc on $\mathbb{Z}_N$
   - $E(a) + E(b) = E(a + b)$ for $a, b \in \mathbb{Z}_N$
   - $aE(b) = E(ab)$ for $a, b \in \mathbb{Z}_N$
2. Define an encryption of poly: $E(f)$
   - $E(a_0 + a_1x + \cdots + a_kx^k) := E(a_0) + E(a_1)x + \cdots + E(a_k)x^k$
3. Can compute $E(fg)$ given $E(f)$ and $E(g)$?
   - Yes if E is ring homomorphic.
4. Can compute $gE(f)$ given $E(f)$ and g?
   - $\left(\sum_i E(a_i)x^i\right)\left(\sum_j b_jx^j\right) = \sum_k\left(\sum_{i+j=k} b_jE(a_i)\right)x^k$
   - where $b_jE(a_i) = E(a_ib_j)$

Private Set Intersection

1. Let f_i be a polynomial corr. to a set A_i with size d
2. Encrypt each f_i into $E(r_i f_i)$ for randomly chosen r_i of degree d
3. $\sum_{i \in I} E(r_i f_i) = E(\sum_{i \in I} r_i f_i) = E(rf)$ for$f = \gcd_i(f_i)$ and a random poly r
4. Decrypt it to get rf and factorize it
5. Random polynomial r has one root at average. How to remove it?
6. What do you learn from rf if we know all r_j, f_j except $j = 1$

Private Set Intersection

1. There are n players P_i with set S_i
2. They want to compute the intersection of S_i without revealing other information
3. Application
   - Several companies collaboratively find their common customers without revealing other information (privacy)
   - In cloud, ...
4. With TTP, it is easy. W/o TTP, use secure multiparty computation, which runs in poly time, but not so practical

Private Set Intersection: Protocol

1. Each party P_i publishes $E(f_i)$
2. All parties collaborate to generate $g_i E(f_i)$ for each i
   - Each party P_j generates a random k-deg poly g_{ij}
   - Publish $g_{ij} E(f_i)$ and compute $g_i E(f_i)$ where $g_i = \sum_{j=1}^{n} g_{ij}$
   - Any $(n-1)$ members can have no information on g_i
3. Group Decryption: $\sum_i g_i E(f_i) = E(\sum_i g_i f_i)$ to $\sum_i g_i f_i$
   - $V(\sum_i g_i f_i) \supset \cap_i S_i$
   - Let $\sum_i g_i f_i = fg$ with $V(f) = \cap_i S_i$
   - g can have a root and it is random. Give a redundancy.
   - e.g. Each set element is encoded to end with 11111

Private Multiset Union

1. Let E be a multiplicative homomorphic encryption on $\mathbb{Z}_n$, i.e. $E(x)E(y) = E(xy)$
2. For a subset $A \subset \wp$, define $M_A = (\prod_{p_i \in A} p_i) \bmod n)$ where p_i's are prime corr. to elements of A
3. $E(M_A)E(M_B) = E(M_A M_B)$
4. Its decryption is a product of the elements in $A \cup B$
5. How to remove the garbage: Use the redundancy function (e.g. each element of $\wp$ ends with 11111.)

Private Set Union [SCK02]

1. How to remove the multiplicities?
2. Represent a set S_i with $1/f_i$ for a polynomial f_i with roots S_i
3. $\sum_i r_i/f_i = r/\mathbf{lcm} f_i$ for a random r
4. How to represent $1/f$ with finite coefficients
5. Rational Reconstruction: Given $f, g \in \mathbb{F}_q[x]/(p(x)$ of degree $\deg(p)/2$, one can efficiently recover f, g from f/g mod $p(x)$

Research Directions

- Non-interactive Set Intersection and Union
- How to merge: Support Intersection and Union with one encryption
  - Set Intersection with AHE
  - Set Union with MHE
- Towards complete set operations ...
- Application in voting, cloud computing, and image processing

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⁎ All pictures are from Google Image