Introduction to moduli spaces

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Part I

What is a moduli space?
Warning : In this talk, there is NO rigorous definition of moduli spaces! We will ignore all technical details.
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Roughly, a moduli space is a (topological, geometric, algebraic) space whose points are in one to one correspondence with geometric objects of one kind.
Toy examples

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$$\{\text{line segment in } \mathbb{R}^2\} \leftrightarrow \{(((x_0, y_0), (x_1, y_1)) | x_0, x_1, y_0, y_1 \in \mathbb{R}\} \cong \mathbb{R}^4$$
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So the moduli space of oriented line segments in $\mathbb{R}^2$ is $\mathbb{R}^4$.

If we don’t want to allow ‘length zero segment’, the moduli space $M$ is $\mathbb{R}^4 - \{((x_0, y_0), (x_0, y_0))\} \cong \mathbb{R}^4 - \Delta \cong \mathbb{R}^4 - \mathbb{R}^2$. 
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$M$ is not only a set, but is a topological space.
Toy examples

Moreover, there is a **universal family** over moduli space $M$. 

Define $U \subset M \times \mathbb{R}^2 \sim \mathbb{R}^4 \times \mathbb{R}^2$ by

$$U = \{ (x_0, y_0, x_1, y_1, (1-t)x_0 + tx_1, (1-t)y_0 + ty_1) | t \in [0,1] \}$$

There is a natural map $\pi: U \to M$ defined by projection to first four coordinates.

Q. Why $U$ is a 'universal family'?

A. For every point $((x_0, y_0), (x_1, y_1)) = p \in M$, $\pi^{-1}(p) = \{ ((1-t)x_0 + tx_1, (1-t)y_0 + ty_1) | t \in [0,1] \} \subset \mathbb{R}^2$, the oriented line segment corresponded $p$!
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the oriented line segment corresponded $p$!
Definition of moduli space

**Definition**

A (fine) **moduli space** of some geometric objects consists of a moduli space $M$, a **universal family** $U$ and a map $\pi : U \to M$ such that

1. there is one to one correspondence between points of $M$ and geometric objects we want to collect.

2. for every point $p \in M$, $\pi^{-1}(p)$ is the corresponded object.
Toy examples

How about non-oriented segments?

In this case, the moduli space of non-oriented line segments in $\mathbb{R}^2$ is
$$M' = (\mathbb{R}^4 - \Delta) / S_2$$

Define an equivalence of oriented line segments as following:
$L_1, L_2 \subset \mathbb{R}^2$ are isomorphic if
$$\exists \text{ a translation } \phi : \mathbb{R}^2 \to \mathbb{R}^2 \text{ such that } \phi(L_1) = L_2.$$ 

Then we can assume the starting point of line segment is the origin. So in this case, the moduli space $M''$ of oriented line segments in $\mathbb{R}^2$ up to translation is
$$M'' = \mathbb{R}^2 - \{(0, 0)\}.$$
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Q. What is the moduli space of unoriented line segments in $\mathbb{R}^2$ up to translation?

A. $(\mathbb{R}^2 - \{0,0\})/S^2$.

Q. What is the moduli space of oriented line segments in $\mathbb{R}^2$ up to isometry?

A. $\mathbb{R}^2 > 0$.

Q. What is the moduli space of oriented line segments in $\mathbb{R}^2$ up to affine transformation?

A. point.

Lesson: Equivalent relations between parameterized objects are very important!
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Lesson : Equivalent relations between parameterized objects are very important!
Fix a vector space $V = \mathbb{R}^n$.

Geometric object want to parameterize: 1-dimensional subspace of $V$. 

Q. What is the moduli space of 1-dimensional subspace of $V$?

Every $v \in V - \{0\}$ determines a unique 1-dimensional subspace $L = \langle v \rangle \subset V$.

$v, v' \in V - \{0\}$ determines the same subspace if $\exists c \in \mathbb{R}^\ast$, $v = cv'$.

$\Rightarrow$ the moduli space of 1-dimensional subspace of $V$ is $(V - \{0\})/\mathbb{R}^\ast = \mathbb{P}(V)$, the projective space!
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$\Rightarrow$ the moduli space of 1-dimensional subspace of $V$ is $(V - \{0\})/\mathbb{R}^* = P(V)$, the projective space!
Moreover, there is the universal family (in this case, universal subspace) $U$ over $P(V)$.

$$U = \{([L], v) | v \in L\} \subset P(V) \times V$$

$\pi : U \rightarrow P(V)$ is the natural projection.
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For $[L] \in P(V)$, $\pi^{-1}([L]) = \{ ([L], v) | v \in L \} \cong L$.

So $P(V)$ with $\pi : U \rightarrow P(V)$ is the fine moduli space of 1-dimensional subspaces of $V$. 
More generally, we can think the moduli space of $k$-dimensional subspaces of $V$ for $1 \leq k \leq n - 1$.
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Exercise : $G(n - 1, V) \cong P(V)$. 
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Exercise: $G(n - 1, V) \cong P(V)$.

Sketch: Fix a positive definite inner product on $V$. Define a map

$$
P(V) \rightarrow G(n - 1, V)$$

$$L \mapsto L^\perp$$

Check this map is bijective.
Some technical issues

If we study moduli spaces in algebraic geometry, there are two important assumptions.

- Usually, we use algebraic closed field $\mathbb{C}$ instead of $\mathbb{R}$.
- We hardly use affine space($\mathbb{C}^n$).
  - Usually we use projective space $\mathbb{P}^n$.
  - It is a compactification of $\mathbb{C}^n$. 
Some famous moduli spaces

$M_g$ : moduli space of nonsingular curves (Riemann surfaces) of genus $g$ up to isomorphism. If $g \geq 2$, it is well-known that the dimension of $M_g$ is $3g - 3$. 

$M_{g,n}$ : moduli space of nonsingular curves of genus $g$ with $n$ distinct points, up to isomorphism. If (1) $g \geq 2$ or (2) $g = 1$ and $n \geq 1$ or (3) $g = 0$ and $n \geq 3$, then the dimension is $3g - 3 + n$.

$M_g(\mathbb{P}^r, d)$ : moduli space of nonsingular curves of genus $g$ in a fixed projective space $\mathbb{P}^r$ with degree $d$.

$C$ : a nonsingular curve of genus $g \geq 2$.

$M(C, r, \alpha)$ : moduli space of stable vector bundles of rank $r$ and first Chern class $\alpha$. 
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Part II

Why we study moduli spaces?
1. Examples of higher dimensional variety

In algebraic geometry, it is extremely difficult to construct an explicit higher dimensional variety.
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For example, in $\mathbb{C}^{1000}$, a zero set of 1,000,000 polynomials defines an algebraic object...

But what is the dimension of it?

Is it smooth? compact? connected? nonempty?
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In many cases, a moduli space of some algebraic objects has an algebraic structure (become variety, scheme, stack...). And there are machineries to get some geometric information of moduli spaces (dimension, smoothness, compactness, ...).
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So moduli spaces gives a plenty of examples of relatively concrete but not obvious higher dimensional algebraic objects.
We recall some classical geometric questions.

**Question**

*How many curves satisfying given conditions are?*
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**Examples:**

- How many lines in plane across given 2 points?
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Examples:

- How many lines in plane across given 2 points?
- How many conics in plane across given 5 points?
2. Answers for classical geometric questions

We recall some classical geometric questions.

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- How many lines in 3-dim space intersect given 4 lines?
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- How many conics in plane across given 5 points?
- How many lines in 3-dim space intersect given 4 lines?

Suprisingly, the answer of last question is neither of 0, 1 nor $\infty$. 
Consider the set of all lines in $\mathbb{P}^3$.
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= the set of all pair of linearly independent homogeneous linear polynomials
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$\equiv$ the set of all pair of linearly independent homogeneous linear polynomials
$\equiv$ the set of all 2-dim subspaces of $V \cong \mathbb{C}^4$
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$\equiv$ Grassmannian $G(2, V)$. 
Consider the set of all lines in $\mathbb{P}^3$

$= \text{the set of all pair of linearly independent homogeneous linear polynomials}$

$= \text{the set of all 2-dim subspaces of } V \cong \mathbb{C}^4$

$= \text{Grassmannian } G(2, V)$.

$U := \{(L, v) \in G(2, V) \times V | v \in L\}$ ⋯ universal family.

$U^* := \{L, v) \in G(2, V) \times V | v \in L, v \neq 0\} \subset U$
Why we study moduli spaces?

Lines in $\mathbb{P}^3$

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Exist two natural maps:

$\pi : U^* \rightarrow G(2, V)$

$(L, v) \mapsto L$

$f : U^* \rightarrow V \rightarrow \mathbb{P}^3$

$(L, v) \mapsto v \mapsto \langle v \rangle$
Why we study moduli spaces?

Lines in $\mathbb{P}^3$

\[ \begin{align*}
U^* & \xrightarrow{f} \mathbb{P}^3 \\
\pi & \downarrow \\
G(2, V) &
\end{align*} \]
Why we study moduli spaces?

Lines in $\mathbb{P}^3$

$U^* \xrightarrow{f} \mathbb{P}^3$

$\pi \downarrow$

$G(2, V)$

$L_i, 1 \leq i \leq 4$: lines in $\mathbb{P}^3$. 

$|_{i=1}^{4} \pi(f^{-1}(L_i))$ is what we want!
Why we study moduli spaces?

Lines in $\mathbb{P}^3$

$U^* \xrightarrow{f} \mathbb{P}^3$

$\pi$

$G(2, V)$

$L_i, 1 \leq i \leq 4$: lines in $\mathbb{P}^3$.

$f^{-1}(L_i)$: set of pairs $(L, v)$ such that $v \in L_i$ and $v \in L$.

$= \text{set of pairs } (L, v) \text{ such that } v \in L \cap L_i.$
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Lines in $\mathbb{P}^3$

$U^* \xrightarrow{f} \mathbb{P}^3$

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$\pi(f^{-1}(L_i))$: set of lines in $\mathbb{P}^3$ meets $L_i$. 
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$\left| \bigcap_{i=1}^{4} \pi(f^{-1}(L_i)) \right|$

is what we want!
Change these things into the language of cohomology:
Cohomology ring of Grassmannian is well-known.
Lines in $\mathbb{P}^3$

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$\sigma_i$: cohomology class corresponded to $\pi(f^{-1}(L_i))$. 
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Cohomology ring of Grassmannian is well-known.

\( \sigma_i \): cohomology class corresponded to \( \pi(f^{-1}(L_i)) \).

In cohomology ring of \( G(2, V) \), \( \sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 \).
Change these things into the language of cohomology:
Cohomology ring of Grassmannian is well-known.

$\sigma_i$: cohomology class corresponded to $\pi(f^{-1}(L_i))$.

In cohomology ring of $G(2, V)$, $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4$.

$$| \bigcap_{i=1}^{4} \pi(f^{-1}(L_i)) | = \int_{G(2,V)} \sigma_1^4 = 2.$$

There exist exactly 2 lines in $\mathbb{P}^3$ meet 4 general lines.
Why we study moduli spaces?

Lines on a Calabi-Yau 3-fold

**Definition**

A quintic threefold is a nonsingular threefold in $\mathbb{P}^4$ defined by single homogeneous equation of degree 5.
Why we study moduli spaces?

Lines on a Calabi-Yau 3-fold

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A **quintic threefold** is a nonsingular threefold in $\mathbb{P}^4$ defined by single homogeneous equation of degree 5.

This is an example of Calabi-Yau threefold appears in string theory.
Why we study moduli spaces?

Lines on a Calabi-Yau 3-fold

Definition

A quintic threefold is a nonsingular threefold in $\mathbb{P}^4$ defined by single homogeneous equation of degree 5.

This is an example of Calabi-Yau threefold appears in string theory.

Question

*How many lines in general quintic threefold?*
Consider the moduli space of lines in $\mathbb{P}^4 : \text{Gr}(2, V)$ where $V \cong \mathbb{C}^5$. 
Consider the moduli space of lines in $\mathbb{P}^4 : Gr(2, V)$ where $V \cong \mathbb{C}^5$. Let $U$ be the universal vector space over $Gr(2, V)$, and let $U^*$ be the complement of zero section.

We have a following diagram

$$
\begin{array}{ccc}
U^* & \xrightarrow{f} & \mathbb{P}^4 \\
\downarrow \pi & & \\
G(2, V). & & 
\end{array}
$$

as before.
Why we study moduli spaces?

Lines on a Calabi-Yau 3-fold

Idea: Make a vector bundle $W$ on $Gr(2, V)$ such that for $L \in Gr(2, V)$, the fiber $W_L$ is the vector space of degree 5 homogeneous polynomials over $L \cong \mathbb{P}^1$.
Why we study moduli spaces?

Lines on a Calabi-Yau 3-fold

Idea: Make a vector bundle $W$ on $Gr(2, V)$ such that for $L \in Gr(2, V)$, the fiber $W_L$ is the vector space of degree 5 homogeneous polynomials over $L \cong \mathbb{P}^1$.

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Moreover, $g_L = 0$ iff $L$ is in the zero set $Z(g) = X$. So we have a section $s$ of $W$, and the number of lines in $X = |Z(s)|$. 

Han-Bom Moon (SNU)
Introduction to moduli spaces
4 August 2010 22 / 35
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$\mathcal{O}(5)$ : a line bundle such that one of section is $g$. 

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Introduction to moduli spaces

4 August 2010 23 / 35
Clemens’ conjecture

Let $X$ be a general quintic threefold. For every $d \in \mathbb{N}$, there exist only finitely many rational curves of degree $d$ on $X$. 

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3. Can learn about the objects parameterized

It is well-known that a nonsingular plane cubic curve has genus 1. Conversely, every nonsingular genus 1 curve is isomorphic to a plane cubic curve.

Moreover, there is an universal equation with a free variable for genus 1 curve. This is a universal equation in the sense that for any genus 1 nonsingular curve $C$, we can find $a \in C$ such that $C$ is isomorphic to plane cubic curve defined by

$$y^2 = x(x-1)(x-a).$$

Except $a \neq 0, 1$, the equation defines a nonsingular genus 1 curve.

For genus 2, the following equation is a universal equation.

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Existence of almost universal equations

Question

For any genus $g$, can we find a universal (or an almost universal) equation (or equations) with almost free variable for genus $g$ curves?

Theorem

If $g \geq 22$, then it is impossible to construct almost universal equations.

This is an immediate corollary of following theorem:

Theorem (Harris-Mumford, Farkas)

For $g \geq 22$, the moduli space $M_g$ is of general type.
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Why we study moduli spaces?

4. Essential elements in modern mathematics and physics

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The Gromov-Witten invariant and quantum cohomology are parts of superstring theory.
Part III

Moduli spaces and birational geometry
Many interesting moduli spaces are not compact.
Compactification of moduli spaces

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Ex) $M_g, M_{g,n}, M_g(\mathbb{P}^r, d), \ldots$
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Example :
1) $\overline{M}_g$ : moduli space of curves of (arithmetic) genus $g$ with nodal singularities which has finite automorphism group.
Compactification of moduli spaces

2) $\overline{M}_{g,n}$ moduli space of curves of (arithmetic) genus $g$ with $n$ distinct smooth points, with nodal singularities which has finite automorphism group.
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3) Compactify $M_g(\mathbb{P}^r, d)$ ⋯ regard $C \subset \mathbb{P}^r$ as an injective morphism $f : C \hookrightarrow \mathbb{P}^r$ from smooth curve $C$ of genus $g$.

$\overline{M}_g(\mathbb{P}^r, d)$ : moduli space of maps from a nodal curve of (arithmetic) genus $g$ to $\mathbb{P}^r$ (called moduli space of stable maps), such that

i) degree of map is $d$,

ii) automorphism group of map is finite.
There might be several different compactifications for one moduli space.
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Fix $\mathcal{A} = (a_1, a_2, \cdots, a_n)$, where $a_i \in \mathbb{Q} \cap (0, 1]$.

$\overline{M}_{g, \mathcal{A}}$ = moduli space of genus $g$ curve $C$ with $n$ smooth points $p_1, p_2, \cdots, p_n$ such that

\begin{enumerate}
  \item each $p_i$ have weight $a_i$.
  \item for $J \subset \{1, 2, \cdots, n\}$, $p_j^1 = p_j^2 = \cdots = p_j^k$ for $j \in J \Rightarrow \sum_{j \in J} a_j \leq 1$
  \item $\omega_C + a_1 p_1 + \cdots + a_n p_n$ is ample. (it guarantees the finiteness of automorphism group.)
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Question

What are the relations between different compactifications?

In our moduli problems, $M_g$, $n$ and $M_g$, $A$, we know that every compactifications have same (open) dense subvariety $M_g$, $n$.

Definition

We say two algebraic varieties $M_1$, $M_2$ are birational if they have open dense subsets $U_1$, $U_2$ respectively, such that $U_1 \sim = U_2$.

1) Is there any morphism $M_g$, $A \to M_g$, $B$ for two different weights $A$, $B$?

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Ultimate goal of algebraic geometry: classify all algebraic varieties.

Consider a nonsingular projective variety.

If $\dim 2$, then for any $X$, by using algebro-geometrical surgery (called blow-up), we can construct a more complicated variety $\tilde{X}$.

But if we know $X$, then we know everything about $\tilde{X}$.

Definition: A nonsingular variety $X$ is called minimal if $X$ is not a blow-up of another variety $X'$.

So we want to classify minimal surfaces.

There are many works about classification of minimal surfaces (Enriques, Kodaira, ...).
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5. If a minimal model exists, there is a unique (log) **canonical model**.
Question

Find log canonical models for given moduli spaces!

Consider $M_{0,n}$. Let $D$ be the subspace corresponded to singular curves.

Theorem (Alexeev-Swinarski, Kiem-M)

Let $\alpha$ be a rational number satisfying $2n - 1 < \alpha \leq 1$. Then the log canonical model $M_{0,n}(\alpha)$ for $\left(M_{0,n}, \alpha D \right)$ satisfies the following:

1. If $2 \cdot \left\lfloor \frac{n}{2} \right\rfloor + 1 < \alpha \leq 1$, then $M_{0,n}(\alpha) \sim_{\text{bir}} M_{0,n}(\alpha)$ where $\alpha = (\epsilon_{\alpha}, \epsilon_{\alpha}, \cdots, \epsilon_{\alpha})$.

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