Stochastic Calculus, Application of Real Analysis in Finance

Workshop for Young Mathematicians in Korea

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Binomial Asset Pricing Model

Assumptions:

- Model stock prices in discrete time
- At each step, the stock price will change to one of two possible values
- Begin with an initial positive stock price $S_0$
- There are two positive numbers, $d$ and $u$, with $0 < d < 1 < u$ such that at the next period, the stock price will be either $dS_0$ or $uS_0$, $d = 1/u$. Of course, real stock price movements are much more complicated.
Binomial Asset Pricing Model

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Binomial Asset Pricing Model

Objectives:

- the definition of probability space and
- the concept of Arbitrage Pricing and its relation to Risk-Neutral Pricing is clearly illuminated

Flip Coins

Toss a coin and when we get a "Head," the stock price moves up, but when we get a "Tail," the price moves down.

$S_1(H) = uS_0$ and $S_1(T) = dS_1$

After the second toss,

$S_2(HH) = uS_1(H) = u^2S_0$, $S_2(HT) = dS_1(H) = duS_0$, $S_2(TH) = uS_1(H) = duS_0$, $S_2(TT) = dS_1(T) = d^2S_0$
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⇒ the price at time 1 by $S_1(H) = uS_0$ and $S_1(T) = dS_1$
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Binomial Asset Pricing Model

For the moment, let us assume that the third toss is the last one and denote by

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

the set of all possible outcomes of the three tosses.
Binomial Asset Pricing Model

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Denote the \( k \)-th component of \( \omega \) by \( \omega_k \).

\( \Rightarrow \) \( S_3 \) depends on all of \( \omega \).

\( \Rightarrow \) \( S_2 \) depends on only the first two components of \( \omega \), \( \omega_1 \) and \( \omega_2 \).
Basic Probability Theory

Probability space \((\Omega, \mathcal{F}, P)\) like \((\mathbb{R}, \mathcal{B}, \mu)\), Lebesgue Measure

- **\(\Omega\)**: the set of all possible realizations of the stochastic economy
  \[
  \Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}
  \]

- **\(\omega\)**: a sample path, \(\omega \in \Omega\)

- **\(\mathcal{F}_t\)**: the sigma field of distinguishable events at time \(t\)
  \[
  \mathcal{F}_2 = \{\emptyset, \{HHH, HHT\}, \{HTH, HTT\}, \{THH, THT\}, \{TTH, TTT\}, \ldots, \Omega\}
  \]

- **\(P\)**: a probability measure defined on the elements of \(\mathcal{F}_t\)
Probability Spaces

Definition
If $\Omega$ is a given set, then a $\sigma$-algebra $\mathcal{F}$ on $\Omega$ is a family $\mathcal{F}$ of subsets of $\Omega$ with the following properties:

- $\emptyset \in \mathcal{F}$
- $F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F}$, where $F^c = \Omega \setminus F$ is the complement of $F$ in $\Omega$
- $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

The pair $(\Omega, \mathcal{F})$ is called a measurable space.
Probability Spaces

Definition
A Probability measure $P$ on a measurable space $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ such that

- $P(\emptyset) = 0$, $P(\Omega) = 1$
- if $A_1, A_2, \ldots \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint then

$$P \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} P(A_i)$$
Random Variable

Definition
If \((\Omega, \mathcal{F}, P)\) is a given probability space, then a function \(Y : \Omega \to \mathbb{R}^n\) is called \(\mathcal{F}\)-measurable if

\[
Y^{-1}(U) := \{\omega \in \Omega; Y(\omega) \in U\} \in \mathcal{F}
\]

for all open sets \(U \in \mathbb{R}^n\) (or, equivalently, for all Borel sets \(U \subset \mathbb{R}^n\)).

In the following we let \((\Omega, \mathcal{F}, P)\) denote a given complete probability space.
A random variable \(X\) is an \(\mathcal{F}\)-measurable function \(X : \Omega \to \mathbb{R}^n\).
Arbitrage Price of Call Options

A money market with interest rate \( r \)

- $1 invested in the money market \( \Rightarrow (1 + r) \) in the next period
- \( d < (1 + r) < u \)
Arbitrage Price of Call Options

A money market with interest rate $r$

- $1$ invested in the money market $\Rightarrow (1 + r)$ in the next period
- $d < (1 + r) < u$

European call option with strike price $K > 0$ and expiration time $1$

- this option confers the right to buy the stock at time $1$ for $K$ dollars
- so is worth $\max[S_1 - K, 0]$
- denote $V_1(\omega) = (S_1(\omega) - K)^+ \triangleq \max[S_1(\omega) - K]$ the value(payoff) at expiration.

$\Rightarrow$ Compute the arbitrage price of the call option at time zero, $V_0$. 
Arbitrage Price of Call Options

Suppose at time zero you sell the call option for $V_0$ dollars.

- $\omega_1 = H \implies$ you should pay off $(uS_0 - K)^+$
- $\omega_1 = T \implies$ you should pay off $(dS_0 - K)^+$

At time 0, we don’t know the value of $\omega_1$. 
Arbitrage Price of Call Options

Suppose at time zero you sell the call option for $V_0$ dollars.

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At time 0, we don’t know the value of $\omega_1$.

Making replicating portfolio:

- $V_0 - \Delta_0 S_0$ dollars in the money market.
- $\Delta_0$ shares of stock

$\Rightarrow$ the value of the portfolio is $V_0$ at time 0.
$\Rightarrow$ the value should be $(S_1 - K)^+$ at time 1.
Arbitrage Price of Call Options

Thus,

\[ V_1(H) = \Delta_0 S_1(H) + (1 + r)(V_0 - \Delta_0 S_0) \quad (1) \]

\[ V_1(T) = \Delta_0 S_1(T) + (1 + r)(V_0 - \Delta_0 S_0) \quad (2) \]
Arbitrage Price of Call Options

Thus,

\[ V_1(H) = \Delta_0 S_1(H) + (1 + r)(V_0 - \Delta_0 S_0) \]  \hspace{1cm} (1)

\[ V_1(T) = \Delta_0 S_1(T) + (1 + r)(V_0 - \Delta_0 S_0) \]  \hspace{1cm} (2)

Subtracting (2) from (1), we obtain

\[ V_1(H) - V_1(T) = \Delta_0 (S_1(H) - S_1(T)), \]  \hspace{1cm} (3)

so that

\[ \Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} \]  \hspace{1cm} (4)
Arbitrage Price of Call Options

Substitute (4) into either (1) or (2) and solve for $V_0$;

$$V_0 = \frac{1}{1 + r} \left[ \frac{1 + r - d}{u - d} V_1(H) + \frac{u - (1 + r)}{u - d} V_1(T) \right]$$

(5)
Arbitrage Price of Call Options

Substitute (4) into either (1) or (2) and solve for $V_0$;

$$V_0 = \frac{1}{1 + r} \left[ \frac{1 + r - d}{u - d} V_1(H) + \frac{u - (1 + r)}{u - d} V_1(T) \right]$$  \hspace{1cm} (5)

Simply,

$$V_0 = \frac{1}{1 + r} [\tilde{p} V_1(H) + \tilde{q} V_1(T)]$$  \hspace{1cm} (6)

where

$$\tilde{p} = \frac{1 + r - d}{u - d}, \quad \tilde{q} = \frac{u - (1 + r)}{u - d} = 1 - \tilde{p}$$

$\Rightarrow$ We can regard them as probabilities of H and T, respectively.

$\Rightarrow$ They are the risk-neutral probabilities.
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Problem - The Movement of Stock Price

A risky investment (e.g. a stock), where the price $X(t)$ per unit at time $t$ satisfies a stochastic differential equation:

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot "noise", \quad (7)$$

where $b$ and $\sigma$ are some given functions.
\[
\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t)W_t
\]
\[ \frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t)W_t \]

Based on many situations, one is led to assume that the "noise", \( W_t \), has these properties:

- \( t_1 \neq t_2 \Rightarrow W_{t_1} \text{ and } W_{t_2} \) are independent.
- \( \{W_t\} \) is stationary, i.e. the (joint)distribution of \( \{W_{t_1+t}, \ldots, W_{t_k+t}\} \) does not depend on \( t \).
- \( \mathbb{E}[W_t] = 0 \) for all \( t \).
\[ \frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t)W_t \]

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- \( \mathbb{E}[W_t] = 0 \) for all \( t \).

Let \( 0 = t_0 < t_1 < \cdots < t_m = t \) and consider a discrete version of (7):

\[ X_{k+1} - X_k = b(t_k, X_k)\Delta t_k + \sigma(t_k, X_k)W_k\Delta t_k, \quad (8) \]

where \( X_j = X(t_j), W_k = W_{t_k}, \Delta t_k = t_{k+1} - t_k. \)
\[ \frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t)W_t \]

Replace \( W_k \Delta_k \) by \( \Delta V_k = V_{t_{k+1}} - V_{t_k} \), where \( \{V_t\}_{t \geq 0} \) is some suitable stochastic process.
\[
\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t)W_t
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Replace \( W_k \Delta_k \) by \( \Delta V_k = V_{t_{k+1}} - V_{t_k} \), where \( \{V_t\}_{t \geq 0} \) is some suitable stochastic process.

It can be easily proved that \( \{V_t\}_{t \geq 0} \) satisfies the four properties which define the standard Brownian Motion, \( \{B_t\}_{t \geq 0} \):
\[
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It can be easily proved that \(\{V_t\}_{t \geq 0}\) satisfies the four properties which define the standard Brownian Motion, \(\{B_t\}_{t \geq 0}\):

- \(B_0 = 0\).
- The increments of \(B_t\) are independent; i.e. for any finite set of times \(0 \leq t_1 < t_2 < \cdots < t_n < T\) the random variables \(B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_n} - B_{t_{n-1}}\) are independent.
- For any \(0 \leq s \leq t < T\) the increment \(B_t - B_s\) has the Gaussian distribution with mean 0 and variance \(t - s\).
- For all \(w\) in a set of probability one, \(B_t(w)\) is a continuous function of \(t\).
\[ \frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t)W_t \]

Thus we put \( V_t = B_t \) and obtain from (8):

\[ X_k = X_0 + \sum_{j=0}^{k-1} b(t_j, X_j)\Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_j)\Delta B_j. \] (9)
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X_k = X_0 + \sum_{j=0}^{k-1} b(t_j, X_j) \Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_j) \Delta B_j. \tag{9}
\]

When \( \Delta t_j \to 0 \), by applying the usual integration notation, we should obtain

\[
X_t = X_0 + \int_0^t b(s, X_s) \, ds + "\int_0^t \sigma(s, X_s) \, dB_s". \tag{10}
\]
\[ \frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t)W_t \]

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When \( \Delta t_j \to 0 \), by applying the usual integration notation, we should obtain

\[ X_t = X_0 + \int_0^t b(s, X_s)ds + "\int_0^t \sigma(s, X_s)dB_s" \] (10)

Now, in the remainder of this chapter we will prove the existence, in a certain sense, of

\[ "\int_0^t f(s, \omega)dB_s(\omega)" \]

where \( B_t(\omega) \) is 1-dim’l Brownian motion.
Construction of the Itô Integral

It is reasonable to start with a definition for a simple class of functions $f$ and then extend by some approximation procedure. Thus, let us first assume that $f$ has the form

$$\phi(t, \omega) = \sum_{j \geq 0} e_j(\omega) \cdot \chi_{[j \cdot 2^{-n}, (j+1)2^{-n})}(t), \quad (11)$$

where $\chi$ denotes the characteristic (indicator) function.
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where $\chi$ denotes the characteristic (indicator) function. For such functions it is reasonable to define

$$\int_0^t \phi(t, \omega) dB_t(\omega) = \sum_{j \geq 0} e_j(\omega)[B_{t_{j+1}} - B_{t_j}](\omega). \quad (12)$$
Construction of the Itô Integral

In general, as we did in Real Analysis, it is natural to approximate a given function \( f(t, \omega) \) by

\[
\sum_{j} f(t_j^*, \omega) \cdot X_{[t_j, t_{j+1})}(t)
\]

where the points \( t_j^* \) belong to the intervals \([t_j, t_{j+1}]\), specifically in Itô Integral, \( t_j^* = t_j \), and then define \( \int_0^t f(s, \omega)dB_s(\omega) \).
Construction of the Itô Integral

Definition
The Itô integral of $f$ is defined by

$$\int_0^t f(s, \omega) dB_s(\omega) = \lim_{n \to \infty} \int_0^t \phi_n(s, \omega) dB_s(\omega) \quad \text{(limit in } L^2(P))$$

(13)

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$E \left[ \int_0^t (f(s, \omega) - \phi_n(s, \omega))^2 dt \right] \to 0 \quad \text{as } n \to \infty. \quad (14)$$

Note that such a sequence $\{\phi_n\}$ satisfying (14) exists.
Example

\[ \int_0^t B_s \, dB_s = \frac{1}{2} B_t^2 - \frac{1}{2} t. \]

(proof)
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(proof)

Put \( \phi_n(s, \omega) = \sum B_j(\omega) \cdot \chi_{[t_j, t_{j+1})}(s) \), where \( B_j = B_{t_j} \). Then
Example

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(proof)

Put \( \phi_n(s, \omega) = \sum B_j(\omega) \cdot X_{[t_j, t_{j+1})}(s) \), where \( B_j = B_{t_j} \). Then

\[
E \left[ \int_0^t (\phi_n - B_s)^2 \, ds \right] = E \left[ \sum_j \int_{t_j}^{t_{j+1}} (B_j - B_s)^2 \, ds \right]
\]

\[ = \sum_j \int_{t_j}^{t_{j+1}} (s - t_j) \, ds \]

\[ = \sum_j \frac{1}{2} (t_{j+1} - t_j)^2 \to 0 \quad \text{as } \Delta t_j \to 0 \]
Example

\[
\int_0^t B_s \, dB_s = \lim_{\Delta t_j \to 0} \int_0^t \phi_n \, dB_s = \lim_{\Delta t_j \to 0} \sum_j B_j \Delta B_j.
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\]

Now

\[
\Delta(B_j^2) = B_{j+1}^2 - B_j^2 = (B_{j+1} - B_j)^2 + 2B_j (B_{j+1} - B_j) = (\Delta B_j)^2 + 2B_j \Delta B_j,
\]
Example

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Now

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and therefore,

\[ B_t^2 = \sum_j \Delta(B_j^2) = \sum_j (\Delta B_j)^2 + 2 \sum_j B_j \Delta B_j \]
Example

\[ \int_{0}^{t} B_s \, dB_s = \lim_{\Delta t_j \to 0} \int_{0}^{t} \phi_n \, dB_s = \lim_{\Delta t_j \to 0} \sum_{j} B_j \Delta B_j. \]

Now

\[ \Delta(B_j^2) = B_{j+1}^2 - B_j^2 = (B_{j+1} - B_j)^2 + 2B_j(B_{j+1} - B_j) \]
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and therefore,

\[ B_t^2 = \sum_{j} \Delta(B_j^2) = \sum_{j} (\Delta B_j)^2 + 2 \sum_{j} B_j \Delta B_j \]

or

\[ \sum_{j} B_j \Delta B_j = \frac{1}{2} B_t^2 - \frac{1}{2} \sum_{j} (\Delta B_j)^2. \]
The Itô Formula

The previous example illustrates that the basic definition of Itô integrals is not very useful when we try to evaluate a given integral as ordinary Riemann integrals without the fundamental theorem of calculus plus the chain rule in the explicit calculations. It turns out that it is possible to establish an Itô integral version of the chain rule, called the Itô formula.
The Itô Formula

Theorem

Let $X_t$ be an Itô process given by

$$dX_t = u dt + v dB_t.$$  

Let $g(t, x) \in C^2([0, \infty) \times \mathbb{R})$. Then $Y_t = g(t, X_t)$ is again an Itô process, and

$$dY_t = \frac{\partial g}{\partial t} (t, X_t) dt + \frac{\partial g}{\partial x} (t, X_t) dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} (t, X_t) \cdot (dX_t)^2, \tag{15}$$

where $(dX_t)^2 = (dX_t) \cdot (dX_t)$ is computed according to the rules

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \quad dB_t \cdot dB_t = dt.$$
Example, Again

Calculate the integral

\[ I = \int_0^t B_s dB_s \]
Example, Again

Calculate the integral

$$I = \int_0^t B_s dB_s$$

Choose $X_t = B_t$ and $g(t, x) = \frac{1}{2}x^2$. Then, $Y_t = \frac{1}{2}B_t^2$. 
Example, Again

Calculate the integral

\[ I = \int_0^t B_s dB_s \]

Choose \( X_t = B_t \) and \( g(t, x) = \frac{1}{2}x^2 \). Then, \( Y_t = \frac{1}{2}B_t^2 \).

By Itô’s formula,

\[
d \left( \frac{1}{2}B_t^2 \right) = B_t dB_t + \frac{1}{2} dt.
\]

In other words,

\[
\frac{1}{2}B_t^2 = \int_0^t B_s dB_s + \frac{1}{2} t.
\]