Estimates of Electrical Fields: Recent Development and Perspective

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Conductivity Equations (free space)

$H$: an entire harmonic function (in particular, $H(x) = a \cdot x$).

\[
\begin{cases}
\nabla \cdot \left( \chi(\mathbb{R}^d \setminus B_1 \cup B_2) + \sum_{j=1}^{2} \sigma_j \chi(B_j) \right) \nabla u = 0 & \text{in } \mathbb{R}^d,

u(x) - H(x) = O(|x|^{1-d}) & \text{as } |x| \to \infty.
\end{cases}
\]

Equivalently,

\[
\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^d \setminus (\partial B_1 \cup \partial B_2),

u|_+ = u|_- & \text{on } \partial B_j, \ j = 1, 2,

\frac{\partial u}{\partial \nu}|_+ = \sigma_j \frac{\partial u}{\partial \nu}|_- & \text{on } \partial B_j, \ j = 1, 2,

u(x) - H(x) = O(|x|^{-1}) & \text{as } |x| \to \infty.
\end{cases}
\]
Conductivity Equations (bounded domain)

\[
\begin{aligned}
\begin{cases}
\nabla \cdot \left( \chi(\Omega \setminus (B_1 \cup B_2)) + \sum_{j=1}^{2} \sigma_j \chi(B_i) \right) \nabla u = 0 & \text{in } \Omega, \\
u = f & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
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Equivalently,

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\begin{cases}
\Delta u = 0 & \text{in } \Omega \setminus (\partial B_1 \cup \partial B_2), \\
u_+ = u_- & \text{on } \partial B_j, \ j = 1, 2, \\
\frac{\partial u}{\partial \nu} = \sigma_j \frac{\partial u}{\partial \nu} & \text{on } \partial B_j, \ j = 1, 2, \\
u = f & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\]
Estimates of $\|\nabla u\|_\infty$

Problem: Estimate $\|\nabla u\|_\infty$ on a bounded region including $B_1$ and $B_2$ as $\epsilon \to 0$ where

$$\epsilon = \text{dist} (B_1, B_2).$$
Estimates of $\| \nabla u \|_\infty$

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$$\epsilon = \text{dist} (B_1, B_2).$$

- If $\sigma_j$ is bounded, then $\| \nabla u \|_\infty(\Omega) \leq C$ regardless of $\epsilon$. (Li-Vogelius ARMA 00, Bonnetier-Vogelius SIMA 00).
- Extended the results to the linear (isotropic) elasticity (Li-Nirenberg CPAM 03).
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Problem: What happens if $\sigma_j$ degenerates to either $\infty$ or 0? (Babuska)
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Problem: What happens if $\sigma_j$ degenerates to either $\infty$ or 0? (Babuska)

Motivation: Estimates of the stress in the composites when grains are very close to each other.
Conductivity Equations ($\sigma_j = 0$ or $\infty$)

$\sigma_j = 0$:

$$\begin{cases}
\Delta u = 0 & \text{in } \Omega \setminus \overline{B_1 \cup B_2}, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_j, \ j = 1, 2, \\
u = f & \text{on } \partial \Omega.
\end{cases}$$
Conductivity Equations ($\sigma_j = 0$ or $\infty$)

$\sigma_j = 0$:

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega \setminus \overline{B_1 \cup B_2}, \\
\left. \frac{\partial u}{\partial n} \right|_+ &= 0 \quad \text{on } \partial B_j, \ j = 1, 2, \\
u &= f \quad \text{on } \partial \Omega.
\end{align*}
\]

$\sigma_j = \infty$:

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega \setminus \overline{B_1 \cup B_2}, \\
u &= \lambda_j \text{(constant)} \quad \text{on } \partial B_j, \ j = 1, 2, \\
u &= f \quad \text{on } \partial \Omega.
\end{align*}
\]

(The constant $\lambda_j$ will be specified.)
Conductivity Equations ($\sigma_j = 0$ or $\infty$)

Suppose $\sigma_1 = \sigma_2 = \sigma$.

- $u^\sigma \rightarrow u^{\sigma_0}$ in $H^1$ as $\sigma_j \rightarrow \sigma_0$ ($\sigma_0 = 0$ or $\infty$) (Friedman-Vogelius, ARMA 88)
Conductivity Equations \((\sigma_j = 0 \text{ or } \infty)\)

Suppose \(\sigma_1 = \sigma_2 = \sigma\).

- \(u^\sigma \to u^{\sigma_0}\) in \(H^1\) as \(\sigma_j \to \sigma_0\) \((\sigma_0 = 0 \text{ or } \infty)\) (Friedman-Vogelius, ARMA 88)

- \(\|u^\sigma - u^{\sigma_0}\|_\infty \leq C \frac{\sigma}{\sigma^2 + 1}\) (K-Seo, IP 99)
Conductivity Equations ($\sigma_j = 0$ or $\infty$)

Suppose $\sigma_1 = \sigma_2 = \sigma$.

- $u^\sigma \to u^{\sigma_0}$ in $H^1$ as $\sigma_j \to \sigma_0$ ($\sigma_0 = 0$ or $\infty$) (Friedman-Vogelius, ARMA 88)

- $\|u^\sigma - u^{\sigma_0}\|_\infty \leq C \frac{\sigma}{\sigma^2 + 1}$ (K-Seo, IP 99)

Question: Does $u^\sigma \to u^{\sigma_0}$ in $W^{1,\infty}$ as $\sigma_j \to \sigma_0$ uniformly in $\epsilon$? ($\partial B_j$ is as smooth as one wishes, say $C^{1,\alpha}$.)
There have been some works (not rigorous) showing that when $\sigma = \infty$

\[ \| \nabla u \|_{L^\infty} \geq \frac{C}{\sqrt{\epsilon}} \]

(Budiansky-Carrier JAM 84, Keller JAM 93, Markenscoff CM 96).

See Milton "The theory of composites" 10.10 for related works.
Circular Inclusions

For $j = 1, 2$, let $B_j = B(Z_j, r_j)$, the disk centered at $Z_j$ and of radius $r_j$. 
Circular Inclusions

For $j = 1, 2$, let $B_j = B(Z_j, r_j)$, the disk centered at $Z_j$ and of radius $r_j$.

Let $R_j, j = 1, 2$, be the reflection with respect to $\partial B_j$, i.e.,

$$R_j(X) := \frac{r_j^2(X - Z_j)}{|X - Z_j|^2} + Z_j, \quad j = 1, 2.$$
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The combined reflection $R_1 R_2$ and $R_2 R_1$ have unique fixed points, say $P_1$ and $P_2$. Let $I$ be the line segment between two fixed points.
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Let $X_j, j = 1, 2$, be the point on $\partial B_j$ closest to the other disk.
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R_j(X) := \frac{r_j^2 (X - Z_j)}{|X - Z_j|^2} + Z_j, \quad j = 1, 2.
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The combined reflection \( R_1 R_2 \) and \( R_2 R_1 \) have unique fixed points, say \( P_1 \) and \( P_2 \). Let \( I \) be the line segment between two fixed points.

Let \( X_j \), \( j = 1, 2 \), be the point on \( \partial B_j \) closest to the other disk.

Let

\[
r_{\text{min}} := \min(r_1, r_2), \quad r_{\text{max}} := \max(r_1, r_2), \quad r_* := \sqrt{\frac{2 r_1 r_2}{r_1 + r_2}}.
\]
Circular Inclusions-lower bounds

**Theorem 1** (Ammari-K-Lim, MA 05). Let \( \epsilon := \text{dist}(B_1, B_2) \) and let \( \nu^{(j)} \) and \( T^{(j)} \), \( j = 1, 2 \), be the unit normal and tangential vector fields to \( \partial B_j \), respectively.

- If \( \sigma > 1 \) and \( \epsilon \) is sufficiently small, there is a constant \( C \) independent of \( \sigma, r_1, r_2, \) and \( \epsilon \) such that

\[
\frac{C_1 \inf_{X \in I} |\langle \nabla H(X), \nu^{(j)}(X_j) \rangle|}{1 - \left( \frac{\sigma - 1}{\sigma + 1} \right)^2 + \left( r_\ast / r_{\text{min}} \right) \sqrt{\epsilon}} \leq |\nabla u|_+(X_j), \quad j = 1, 2,
\]

- If \( \sigma < 1 \), then

\[
\frac{C_1 \inf_{X \in I} |\langle \nabla H(X), T^{(j)}(X_j) \rangle|}{1 - \left( \frac{\sigma - 1}{\sigma + 1} \right)^2 + \left( r_\ast / r_{\text{min}} \right) \sqrt{\epsilon}} \leq |\nabla u|_+(X_j), \quad j = 1, 2.
\]
Circular Inclusions-upper bound

**Theorem 2** (Ammari-K-Lee-Lee-Lim, JMPA 07). Let $\Omega$ be a bounded set containing $B_1$ and $B_2$. Then there is a constant $C$ independent of $\sigma$, $r_1$, $r_2$, $\epsilon$ such that

$$\|\nabla u\|_{L^\infty(\Omega)} \leq \frac{C_2 \|\nabla H\|_{L^\infty(\Omega)}}{1 - \left(\frac{\sigma - 1}{\sigma + 1}\right) + \left(\frac{r_*}{r_{\text{max}}}\right) \sqrt{\epsilon}}.$$
Theorem 3 (Ammari-K-Lee-Lee-Lim, JMPA 07). Let $\Omega$ be a bounded set containing $B_1$ and $B_2$. Then there is a constant $C$ independent of $\sigma, r_1, r_2, \varepsilon$ such that

$$
\|\nabla u\|_{L^\infty(\Omega)} \leq \frac{C_2 \|\nabla H\|_{L^\infty(\Omega)}}{1 - \left(\frac{\sigma - 1}{\sigma + 1}\right) + \left(r_*/r_{\text{max}}\right)\sqrt{\varepsilon}}.
$$

If $\sigma > 1, r = r_1 = r_2$, and $H = a \cdot x$, then

$$
\frac{C_1 |\langle a, \nu^{(j)}(X_j)\rangle|}{\frac{1}{\sigma} + \sqrt{\frac{\varepsilon}{r}}} \leq \|\nabla u\|_{L^\infty(\Omega \setminus (B_1 \cup B_2))} \leq \frac{C_2 |a|}{\frac{1}{\sigma} + \sqrt{\frac{\varepsilon}{r}}}.
$$
**Theorem 4** (Ammari-K-Lee-Lee-Lim, JMPA 07). Let $\Omega$ be a bounded set containing $B_1$ and $B_2$. Then there is a constant $C$ independent of $\sigma$, $r_1$, $r_2$, $\epsilon$ such that

$$\|\nabla u\|_{L^\infty(\Omega)} \leq \frac{C_2\|\nabla H\|_{L^\infty(\Omega)}}{1 - \left(\frac{\sigma-1}{\sigma+1}\right) + \left(r_*/r_{\text{max}}\right)\sqrt{\epsilon}}.$$ 

If $\sigma > 1$, $r = r_1 = r_2$, and $H = a \cdot x$, then

$$\frac{C_1|\langle a, \nu^{(j)}(X_j)\rangle|}{\frac{1}{\sigma} + \sqrt{\frac{\epsilon}{r}}} \leq \|\nabla u\|_{L^\infty(\Omega \setminus (B_1 \cup B_2))} \leq \frac{C_2|a|}{\frac{1}{\sigma} + \sqrt{\frac{\epsilon}{r}}}.$$ 

If $r \approx \epsilon$, No blow-up.
Convex Inclusions in 2D

The solution $u$ is represented as

$$u(x) = H(x) + S_{B_1}[\varphi_1](x) + S_{B_2}[\varphi_2](x), \quad x \in \mathbb{R}^2,$$

where

$$\varphi_1 = \frac{2(\sigma - 1)}{\sigma + 1} \sum_{m=0}^{\infty} \left( \frac{\sigma - 1}{\sigma + 1} \right)^m \frac{\partial}{\partial \nu} \left[ (R_2 R_1)^m (I - \frac{\sigma - 1}{\sigma + 1} R_2) H \right] \bigg|_{\partial B_1},$$

$$\varphi_2 = \frac{2(\sigma - 1)}{\sigma + 1} \sum_{m=0}^{\infty} \left( \frac{\sigma - 1}{\sigma + 1} \right)^m \frac{\partial}{\partial \nu} \left[ (R_1 R_2)^m (I - \frac{\sigma - 1}{\sigma + 1} R_1) H \right] \bigg|_{\partial B_2},$$
Inclusions close to the boundary

$B$ and $\Omega$ are disks of radius $r$ and $\rho$ such that $B \subseteq \Omega$. 
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$$\epsilon := \text{dist}(B, \partial \Omega), \quad r^* := \sqrt{\frac{\rho - r}{\rho r}}$$
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$$
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$$

Dirichlet problem

$$
\begin{cases}
\nabla \cdot \left( \chi(\Omega \setminus \overline{B}) + \sigma \chi(B) \right) \nabla u = 0 & \text{in } \Omega, \\
u = f & \text{on } \partial \Omega.
\end{cases}
$$
Theorem 5 (AKLLL, JMPA 07). (i) If $\sigma > 1,$
\[
\frac{C_1 \inf_{X \in J_1} |\langle \nabla D_\Omega(f)(X), \nu_B(X_1) \rangle|}{1/\sigma + 4r^* \sqrt{\epsilon}} \leq |\nabla u|_+(X_1),
\]
and
\[
\frac{C_1 \inf_{X \in J_2} |\langle \nabla D_\Omega(f)(X), \nu_\Omega(X_2) \rangle|}{1/\sigma + 4r^* \sqrt{\epsilon}} \leq |\nabla u|_-(X_2).
\]
Here $\nu_B$ and $\nu_\Omega$ denote the outward unit normal to $\partial B$ and $\partial \Omega.$

(ii)
\[
\|\nabla u\|_{L^\infty(\Omega)} \leq \frac{C_2 \|f\|_{C^{1,\alpha}(\partial \Omega)}}{1/\sigma + r^* \sqrt{\epsilon}}.
\]

Inclusions close to the boundary (Dirichlet problem)
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**Theorem 6** (AKLLL, JMPA 07). (i) If $\sigma > 1$,

\[
\begin{aligned}
C_1 \inf_{X \in J_1} \frac{|\langle \nabla D_\Omega (f)(X), \nu_B (X_1) \rangle|}{1/\sigma + 4 r^* \sqrt{\epsilon}} \leq |\nabla u|_+ (X_1), \\
\end{aligned}
\]

and

\[
\begin{aligned}
C_1 \inf_{X \in J_2} \frac{|\langle \nabla D_\Omega (f)(X), \nu_\Omega (X_2) \rangle|}{1/\sigma + 4 r^* \sqrt{\epsilon}} \leq |\nabla u|_- (X_2).
\end{aligned}
\]

Here $\nu_B$ and $\nu_\Omega$ denote the outward unit normal to $\partial B$ and $\partial \Omega$.

(ii)

\[
\| \nabla u \|_{L^\infty (\Omega)} \leq \frac{C_2 \| f \|_{C^{1,\alpha} (\partial \Omega)}}{1/\sigma + r^* \sqrt{\epsilon}}.
\]

\[
\frac{A}{1/\sigma + r^* \sqrt{\epsilon}} \leq \| \nabla u \|_{L^\infty (\Omega)} \leq \frac{B}{1/\sigma + r^* \sqrt{\epsilon}},
\]
Inclusions close to the boundary (Neuman problem)

\[ \begin{aligned}
\n\n& \nabla \cdot (1 + (k - 1)\chi(B))\nabla u = 0 \quad \text{in } \Omega, \\
& \frac{\partial u}{\partial \nu} = g \quad \text{on } \partial\Omega.
\end{aligned} \]
Inclusions close to the boundary (Neuman problem)

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\end{aligned} \]

**Theorem 8** (AKLLL, JMPA 07).

(i) If \( \sigma < 1 \), then

\[
C_1 \inf_{X \in J_1} \frac{\langle \nabla S_{\Omega}(g)(X), T_B(X_1) \rangle}{\sigma + 4r^* \sqrt{\epsilon}} \leq |\nabla u|_+(X_1),
\]

and

\[
C_1 \inf_{X \in J_2} \frac{\langle \nabla S_{\Omega}(g)(X), T_{\Omega}(X_2) \rangle}{\sigma + 4r^* \sqrt{\epsilon}} \leq |\nabla u|_-(X_2).
\]

Here \( T_B \) and \( T_{\Omega} \) denote the positively oriented unit tangent vector field on \( \partial B \) and \( \partial \Omega \), respectively.

(ii)

\[
\| \nabla u \|_{L^\infty(\Omega)} \leq \frac{C_2 \| g \|_{C^\alpha(\partial \Omega)}}{\sigma + r^* \sqrt{\epsilon}}.
\]
Theorem 9 (Yun, SIAP 07). \( B_1 \) and \( B_2 \) are convex ‘with proper locations’, \( \sigma = \infty \), and 
\( H(x, y) = x \). Then,

\[
|u|_{\partial B_1} - u|_{\partial B_2} \geq C_1 \sqrt{\epsilon}, \quad \|\nabla u\|_{L^\infty} \leq \frac{C_2}{\sqrt{\epsilon}}.
\]
Perfect conductors in 3D

$B_1, B_2$: convex perfect conductors ($\sigma = \infty$)

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega \setminus \overline{B_1 \cup B_2}, \\
u = \lambda_j(\text{constant}) & \text{on } \partial B_j, \ j = 1, 2, \\
u = f & \text{on } \partial \Omega.
\end{cases}
\]
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\end{cases}
\]

Additional conditions:

\[
\int_{\partial B_j} \frac{\partial u}{\partial \nu} \left. d\sigma = 0.\right)
\]
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\[ \begin{cases} 
\Delta u = 0 & \text{in } \Omega \setminus \overline{B_1 \cup B_2}, \\
\lambda_j (\text{constant}) & \text{on } \partial B_j, \ j = 1, 2, \\
u = f & \text{on } \partial \Omega.
\end{cases} \]

Additional conditions:

\[ \int_{\partial B_j} \frac{\partial u}{\partial \nu} \bigg|_{+} \ d\sigma = 0. \]

\[ |\nabla u(x)| \geq \frac{C}{\epsilon} |\lambda_1 - \lambda_2|. \]
Perfect conductors in 3D

\[
\begin{align*}
\Delta v_1 &= 0 \quad \text{in } \Omega \setminus \overline{B_1 \cup B_2}, \\
u &= 1 \quad \text{on } \partial B_1, \\
u &= 0 \quad \text{on } \partial B_2, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

\[
\begin{align*}
\Delta v_2 &= 0 \quad \text{in } \Omega \setminus \overline{B_1 \cup B_2}, \\
u &= 0 \quad \text{on } \partial B_1, \\
u &= 1 \quad \text{on } \partial B_2, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

\[
\begin{align*}
\Delta v_0 &= 0 \quad \text{in } \Omega \setminus \overline{B_1 \cup B_2}, \\
u &= 0 \quad \text{on } \partial B_1, \\
u &= 0 \quad \text{on } \partial B_2, \\
u &= f \quad \text{on } \partial \Omega.
\end{align*}
\]
Perfect conductors in 3D

\[
\begin{aligned}
\Delta v_1 &= 0 \quad \text{in } \Omega \setminus \overline{B_1 \cup B_2}, \\
u &= 1 \quad \text{on } \partial B_1, \\
u &= 0 \quad \text{on } \partial B_2, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

\[
\begin{aligned}
\Delta v_2 &= 0 \quad \text{in } \Omega \setminus \overline{B_1 \cup B_2}, \\
u &= 0 \quad \text{on } \partial B_1, \\
u &= 1 \quad \text{on } \partial B_2, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

\[
\begin{aligned}
\Delta v_0 &= 0 \quad \text{in } \Omega \setminus \overline{B_1 \cup B_2}, \\
u &= 0 \quad \text{on } \partial B_1, \\
u &= 0 \quad \text{on } \partial B_2, \\
u &= f \quad \text{on } \partial \Omega.
\end{aligned}
\]

\[u = \lambda_1 v_1 + \lambda_2 v_2 + v_0.\]
Perfect conductors in 3D

\[ \lambda_1 \int_{\partial B_1} \frac{\partial v_1}{\partial \nu} + \lambda_2 \int_{\partial B_1} \frac{\partial v_2}{\partial \nu} + \int_{\partial B_1} \frac{\partial v_0}{\partial \nu} = 0, \]

\[ \lambda_1 \int_{\partial B_2} \frac{\partial v_1}{\partial \nu} + \lambda_2 \int_{\partial B_2} \frac{\partial v_2}{\partial \nu} + \int_{\partial B_1} \frac{\partial v_0}{\partial \nu} = 0. \]
Perfect conductors in 3D

\[\lambda_1 \int_{\partial B_1} \frac{\partial v_1}{\partial \nu} + \lambda_2 \int_{\partial B_1} \frac{\partial v_2}{\partial \nu} + \int_{\partial B_1} \frac{\partial v_0}{\partial \nu} = 0,\]

\[\lambda_1 \int_{\partial B_2} \frac{\partial v_1}{\partial \nu} + \lambda_2 \int_{\partial B_2} \frac{\partial v_2}{\partial \nu} + \int_{\partial B_1} \frac{\partial v_0}{\partial \nu} = 0.\]

Suppose \(\partial B_j\) is given locally by

\[x_d = \psi_1(x') + \frac{\epsilon}{2}\] and \[x_d = -\psi_2(x') - \frac{\epsilon}{2}\]

\(|x'| < \delta\).

Then, the blow-up rate is "determined" by

\[\int_{|x'| < \delta} \frac{1}{\psi_1(x') + \psi_2(x') + \epsilon} dx'.\]

(Bao-Li-Yin, ARMA to appear)
Perfect conductors in 3D

**Theorem 10** (Bao-Li-Yin, ARMA to appear). *If* $\partial B_1$ *and* $\partial B_2$ *are strictly convex, then*

$$
\|\nabla u\|_{L^\infty} \leq C' \|f\|_{C^2(\partial \Omega)} \left\{ \begin{array}{ll}
\frac{1}{\sqrt{\epsilon}}, & d = 2, \\
\frac{1}{\epsilon |\ln \epsilon|}, & d = 3, \\
\frac{1}{\epsilon}, & d \geq 4,
\end{array} \right.
$$

*and*

$$
\|\nabla u\|_{L^\infty} \geq C |Q[f]| \left\{ \begin{array}{ll}
\frac{1}{\sqrt{\epsilon}}, & d = 2, \\
\frac{1}{\epsilon |\ln \epsilon|}, & d = 3, \\
\frac{1}{\epsilon}, & d \geq 4,
\end{array} \right.
$$

*where*

$$
Q[f] = \int_{\partial B_1} \frac{\partial v_0}{\partial \nu} \int_{\partial \Omega} \frac{\partial v_2}{\partial \nu} - \int_{\partial B_2} \frac{\partial v_0}{\partial \nu} \int_{\partial \Omega} \frac{\partial v_1}{\partial \nu}.
$$
Perfect conductors in 3D

If

\[ \psi_1(x') + \psi_2(x') \approx |x'|^{2m}, \quad m > 1, \]

then the blow-up rate is

\[ \epsilon^{-1/2m}(2D), \quad \epsilon^{-1/m}(3D). \]
Perfect conductors in 3D

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$$\psi_1(x') + \psi_2(x') \approx |x'|^{2m}, \quad m > 1,$$

then the blow-up rate is

$$\epsilon^{-1/2m}(2D), \quad \epsilon^{-1/m}(3D).$$

Flat inclusions have lower blow-up rate!
Perfect conductors in 3D

A work of Lim-Yun ($\sigma = \infty$): $B_1$ and $B_2$ are balls in $\mathbb{R}^d$. Let $h$ be a solution to

$$\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^d \setminus \overline{B_1 \cup B_2}, \\
u(x) = O(|x|^{1-d}) & \text{as } |x| \to \infty, \\
u = k_j \text{ (constant)} & \text{on } \partial B_j, \ j = 1, 2, \\
\int_{\partial B_1} \frac{\partial h}{\partial \nu} d\sigma = 1, \quad \int_{\partial B_1} \frac{\partial h}{\partial \nu} d\sigma = -1.
\end{cases}$$
Perfect conductors in 3D

A work of Lim-Yun ($\sigma = \infty$): $B_1$ and $B_2$ are balls in $\mathbb{R}^d$. Let $h$ be a solution to

$$\begin{cases}
\Delta u = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B_1 \cup B_2}, \\
u(x) = O(|x|^{1-d}) \quad \text{as } |x| \to \infty, \\
u = k_j \text{ (constant)} \quad \text{on } \partial B_j, \ j = 1, 2, \\
\int_{\partial B_1} \frac{\partial h}{\partial \nu} d\sigma = 1, \quad \int_{\partial B_1} \frac{\partial h}{\partial \nu} d\sigma = -1.
\end{cases}$$

$$\lambda_1 - \lambda_2 = \int_{\partial B_1} \frac{\partial h}{\partial \nu} H d\sigma - \int_{\partial B_2} \frac{\partial h}{\partial \nu} H d\sigma.$$
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Lim-Yun constructed such an $h$ when $B_1$ and $B_2$ are balls in $\mathbb{R}^d$ and obtain the same result for balls (independent work).
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In 2D,

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$$
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Thank You!