Circle Actions on Symplectic Manifolds

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Introduction

Def. Symplectic Manifold
Symplectic manifold \((M, \omega)\) is a \(2n\)-dimensional manifold with the differential 2-form \(\omega\) which is closed and non-degenerate.

Def. Symplectic action
Let \(G\) be a Lie group and \((M, \omega)\) be a symplectic manifold. We say that \(G\) acts on \(M\) symplectically if \(G\) preserves the symplectic form \(\omega\), i.e.

\[ g^* \omega = \omega \]

for \(\forall g \in G\). In this case, we call \((M, \omega)\) is a symplectic \(G\)-manifold.
**Def. Hamiltonian action**

Let \((M, \omega)\) be a symplectic \(G\)-manifold. We say that the \(G\)-action is hamiltonian if

\[
\iota_X \omega \text{ is exact for } \forall X \in T_e G
\]

Equivalently, the \(G\)-action is hamiltonian if there exists a map \(\mu : M \to \mathfrak{g}^*\) such that

\[
d < \mu, X > = \iota_X \omega, \forall X \in T_e G.
\]

Here, we call \(\mu\) moment map.
Introduction

Main Theorem

properties of \( \langle \mu, X \rangle \)

1. A subgroup of \( G \) generated by \( X \) preserves a levels of \( \langle \mu, X \rangle \).
2. \( \langle \mu, X \rangle \) is a Morse function such that \( M^X = \text{Crit } f \).
3. If \( X \) generates a circle, for \( z \in M^X \), \( \text{ind } (z) \) equals to \( 2i_z \) where \( i_z \) is a number of negative weights of \( S^1 \)-representation at \( z \).
Example

Consider \((\mathbb{P}^1, \omega)\) with the normalized symplectic form with a \(S^1\)-action given by \(t \cdot [z_0, z_1] = [t z_0, z_1]\). The moment map is a height function

\[
\mu([z_0, z_1]) = \frac{|z_0|^2}{|z_0|^2 + |z_1|^2}
\]

with \(\text{Max } \mu - \text{Min } \mu = \int_{\mathbb{P}^1} \omega = 1\)

\[
\begin{array}{c}
\text{Diagram}
\end{array}
\]
Def. Generalized moment map

Consider a symplectic manifold \((M, \omega)\) with a symplectic \(S^1\)-action. Assume that \(\omega\) represents an integral cohomology class in \(H^2(M)\). Then, the generalized moment map \(\mu : M \to S^1 \equiv \mathbb{R}/\mathbb{Z}\) is defined by

\[
\mu(x) = \int_{x_0}^x \iota_X \omega (\text{mod } \mathbb{Z})
\]

for fixed \(x_0\) and \(X \equiv 1\) in \(T_eS^1 \equiv \mathbb{R}\).

Remark

Locally, the differential of the generalized moment map \(d\mu = \iota_X \omega\). So, if \(M\) is simply connected, then \(\mu\) can be lifted to a \(\mathbb{R}\)-valued function, i.e. the given action must be hamiltonian. But, the converse is not true in general.
T.Frankel, 1959

Let \((M, \omega)\) be a Kähler manifold with a \(\omega\)-preserving circle action. Then, the fixed set \(M^{S^1}\) is non-empty if and only if it is hamiltonian.

D.McDuff, 1988

1. Let \((M, \omega)\) be a 4-dimensional symplectic manifold with a symplectic circle action. Then, the fixed set \(M^{S^1}\) is non-empty if and only if it is hamiltonian.

2. There exists a circle action on some 6-dimensional symplectic manifold with non-empty fixed set which is not hamiltonian.
S. Tolman and J. Weitsman, 1999

Let \((M, \omega)\) be a \(2n\)-dimensional symplectic manifold with a semifree symplectic circle action with isolated fixed set \(M^{S^1}\). Then, \(M^{S^1}\) is non-empty if and only if it is hamiltonian. Moreover, \(H^*(M; \mathbb{Z})\) and \(H_{S^1}^*(M; \mathbb{Z})\) is equal to the one of \(\mathbb{P}^1 \times \ldots \times \mathbb{P}^1\) with the diagonal action.

Frankel-McDuff Conjecture

Let \((M, \omega)\) be a \(2n\)-dimensional symplectic manifold with a symplectic circle action with isolated fixed set \(M^{S^1}\). Then, it is hamiltonian if and only if \(M^{S^1} \neq \emptyset\).
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Some Remarks

1. Let $\pi : E \to B$ a $G$-vector bundle. Then, its equivariant characteristic classes $ch^G$ are defined by

$$ch^G(E) := ch(E \times_G EG) \in H^*_G(M).$$

2. Let $M$ be a compact $G$-manifold and let $F$ be a fixed component. Then $e^G(\nu_F)$ is invertible in $S^{-1} H^*_G(F) \equiv S^{-1} H^*(F) \otimes H^*(BG)$, where $\nu_F$ is a normal bundle over $F$ in $M$ and $S$ is the multiplicative set $H^*(BG) - (0)$.

3. For any $u \in H^*_G(M)$, There is a map

$$\int_M : H^*_G(M) \to H^*(BG),$$

which is called "integration along the fiber"
Atiyah-Bott-Berline-Vergne Localization formula (ABBV)

**Theorem (ABBV)**

Let $T$ be a torus acting on compact manifold $M$. For any class $u \in H_T^*(M)$,

$$\int_M u = \sum_F \int_F \frac{u|_F}{e^T(\nu_F)}$$

where $u|_F$ is the restriction of $u$ to $F$, $e^T$ is an equivariant euler class, and $\nu_F$ is a normal bundle over fixed component $F$. 
Main Theorems

**Theorem 1**

Let \((M, \omega)\) be a 6-dimensional symplectic \(S^1\)-manifold. Then, if the fixed components are 2-spheres and if none of the weights equal to \(\pm 1\), it is hamiltonian.

**Theorem 2**

Let \((M, \omega)\) be a \(2n\)-dimensional symplectic \(T^{n-1}\)-manifold. If there exists a circle subgroup \(S^1\) fixing some component \(Z\) with \(\chi(Z) \neq 0\), given \(T^{n-1}\) action is hamiltonian.
Proof of theorem 1

Assume that there is no fixed sphere with index 0 and 4. Let $c_1^{S^1}$ be a equivariant first chern class. By the localization formula,

$$0 = \int_M 1 = \sum_F \int_F \frac{1 \mid F}{e^{S^1(\nu_F)}}.$$

If $F$ is a 2-sphere, $\nu_F$ is a direct sum of two line bundles $E^F_p \oplus E^F_{-q}$, where $(p, -q)$ is weights of tangential $S^1$-representation on $F$ and $E^F_p$ ($E^F_{-q}$ resp.) is a subbundle of $\nu_F$ on which $S^1$ acts as a $p$-times rotation ($-q$ resp.), $p, q > 0$. Denote $k^F$ be the first chern number of $E^F_p$ and $l^F$ be the one of $E^F_{-q}$.
Let $u$ be a generator of $H^*(BS^1)$ and $x$ be a generator of $H^*(F)$. Assume $p, q \neq \pm 1$.

1. \(e^{S^1}(\nu_F) = (pl^F - qk^F)xu - pqu^2\).

2. \((e^{S^1})^{-1} = \frac{(pl^F - qk^F)x + pqu}{-p^2q^2u^3}\)

3. A 4-dimensional symplectic $S^1$-manifold with two fixed spheres is a Hirzebruch surface.

4. $\mathbb{Z}_p (\mathbb{Z}_q \text{ resp.})$-fixed component is 4-dimensional and it has two fixed spheres, so it should be a Hirzebruch surfaces $H_p (H_{-q} \text{ resp.})$. Moreover, $F$ is a minimum(maximum resp.) of $H_p (H_{-q})$ w.r.t the moment map.

5. For any Hirzebruch surface with the moment map, its minimum sphere(maximum resp.) has the non-negative(non-positive resp.) euler number. So, $k^F \geq 0$ and $l^F \leq 0$. So, $pl^F - qk^F \leq 0$. 
\[ 0 = \int_M 1 = \sum_F \int_F \frac{1}{e^{S^1}(\nu_F)} = \sum_F \int_F \frac{(pl^F - qk^F)x + pqu}{-p^2q^2u^3} \]
\[ = \sum_F \frac{(pl^F - qk^F)}{-p^2q^2u^3} \geq 0. \]

So, it should be zero. equivalently, all \( l^F \) and \( k^F \) should be zeros.

\[ 0 = \int_M c_1^{S^1} = \sum_F \int_F \frac{c_1^{S^1}|_F}{e^{S^1}(\nu_F)}. \]

1. \[ c_1^{S^1}|_F = (p - q)u + (k^F + l^F + 2)x \]
2. \[ \frac{c_1^{S^1}|_F}{e^{S^1}(\nu_F)} = \frac{((pl^F - qk^F)(p - q) + pq(k+l+2))ux}{-p^2q^2u^3} \]
So, if all $l^F$ and $k^F$ are zero, then

$$0 = \int_M c_1^{S^1} = \sum_F \int_F \frac{c_1^{S^1}|F}{e^{S^1}(\nu_F)}.$$ 

$$= \sum_F \frac{((pl^F - qk^F)(p - q) + pq(k + l + 2))ux}{-p^2q^2u^3}$$ 

$$= \sum_F \frac{2pqu}{-p^2q^2u^3} < 0$$

So, it is a contradiction.
Result of M.Kim

Let \((M, \omega)\) be a 2\(n\)-dimensional symplectic \(T^{n-1}\)-manifold with \(M^{T^{n-1}} \neq \emptyset\). Then the action is hamiltonian.

Proof of theorem 2

Let \((M, \omega)\) be a 2\(n\)-dimensional symplectic \(T^{n-1}\)-manifold. Let \(H_1 = S^1 \subset T^{n-1}\) be a subgroup having some fixed component \(Z\) such that \(\chi(Z) \neq 0\). Decompose \(T^{n-1}\) into \(H_1 \times H_2 \times \ldots \times H_{n-1}\), where \(H_i\) is a circle in \(T^{n-1}\). Then,

1. \(H_i\) acts on \(Z\). (\(\because \forall z \in Z, \forall t_i \in H_i, t_i \cdot z\) is fixed by \(H_1\)).
2. 
   \[
   \chi(Z) = \sum_{F \subset Z^{H_i}} \chi(F) \neq 0
   \]

   , by Atiyah-Hirzebruch formula. So, \(M^{H_1 \times H_i} \neq \emptyset\). By induction, \(M^{T^{n-1}} \neq \emptyset\).

By M.Kim’s result, it should be hamiltonian.