Minkowski sum
and simple polytopes

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Minkowski sum
(Hermann Minkowski, 1864–1909)

Let $M_1$ and $M_2$ be subsets in $\mathbb{R}^n$.

**Definition.** The *Minkowski sum* is the set
\[ \{ x \in \mathbb{R}^n : x = x_1 + x_2, \ x_1 \in M_1, \ x_2 \in M_2 \} . \]

**Lemma.** If $M_1$ and $M_2$ are convex polytopes then $M_1 + M_2$ is again convex polytope.

The collection of all convex polytopes in $\mathbb{R}^n$ is denoted by $\mathcal{M}_n$.

The Minkowski sum gives an abelian monoid structure on $\mathcal{M}_n$, where zero $0$ is the point $0 = (0, \ldots , 0) \in \mathbb{R}^n$.

- There is the canonical homomorphism $\mathbb{R}^n \to \mathcal{M}_n$: image of a vector $v \in \mathbb{R}^n$ is the one point polytope.
- $\mathcal{M}_n$ has the structure of $\mathbb{R}$-module:
  for given $\lambda \in \mathbb{R}$ and $M \in \mathcal{M}_n$
  \[ \lambda M = \{ \lambda x \in \mathbb{R}^n, \ x \in M \} . \]
- In $\mathcal{M}_n$ we have
  \[ M_1 + M = M_2 + M \Rightarrow M_1 = M_2 . \]
Denote by $\text{conv}(v_1, \ldots, v_N)$ the convex hull of the points $v_1, \ldots, v_N$ in $\mathbb{R}^n$.

**Lemma.** $\text{conv}(v_1, \ldots, v_k) + \text{conv}(w_1, \ldots, w_l) =$

$= \text{conv}(v_1 + w_1, \ldots, v_i + w_j, \ldots, v_k + w_l)$.

**Proof.** Set $M_1 = \text{conv}(v_1, \ldots, v_k)$ and $M_2 = \text{conv}(w_1, \ldots, w_l)$. Then $v_i + w_j \in M_1 + M_2$ for any $1 \leq i \leq k$ and $1 \leq j \leq l$. Take

$$x_1 = \sum_{i=1}^{k} t_i v_i, \quad x_2 = \sum_{j=1}^{l} \tau_j w_j,$$

where $\sum_{i=1}^{k} t_i = 1, \ t_i \geq 0$, and $\sum_{j=1}^{l} \tau_j = 1, \ \tau_j \geq 0$. Then

$$x_1 + x_2 = \sum_{i=1}^{k} t_i \left( \sum_{j=1}^{l} \tau_j \right) v_i + \sum_{j=1}^{l} \tau_j \left( \sum_{i=1}^{k} t_i \right) w_j =$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{l} \xi_{ij} (v_i + w_j),$$

where $\xi_{ij} = t_i \tau_j \geq 0$ and $\sum_{i=1}^{k} \sum_{j=1}^{l} \xi_{ij} = 1$. 

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**Minkowski sums in** $\mathbb{R}^1$

**Examples.**

$M_1 = [-1, 2] \subset \mathbb{R}^1, \quad M_2 = [1, 2] \subset \mathbb{R}^1.$

$M_1 + M_1 = [-2, 4] = 2[M_1],$

$M_1 - M_1 = [-1, 2] + [-2, 1] = [-3, 3].$

$M_1 + M_2 = [0, 4], \quad M_1 - M_2 = [-3, 1].$

**General case:**

$M_i = [a_i, b_i], \ a_i \leq b_i, \ i = 1, 2.$

$M_1 + M_2 = [a, b], \text{ where } a = a_1 + a_2, \ b = b_1 + b_2.$

$M_1 - M_2 = [a', b'], \text{ where } a' = a_1 - b_2, \ b' = b_1 - a_2.$

So,

$M_1 - M_1 = (b_1 - a_1)[-1, 1].$
Minkowski sums in $\mathbb{R}^2$

$M_1 + M_2$ is the convex hall of the set $\{(1, 0), (0, 1), (2, 0), (1, 1)\}$.

Set $e_0 = (0, 0), e_1 = (1, 0), e_2 = (0, 1)$. Let $M_1 = \text{conv}(e_0, e_1, e_2)$ and $M_2 = \text{conv}(e_0, x)$, where $x \in \mathbb{R}^2$. Then

$$M_1 + M_2 = \text{conv}(e_0, e_1, e_2, x, e_1 + x, e_2 + x).$$

Thus we obtain

$$5 - \text{gon}, \text{ if } x = e_1 + e_2$$

and

$$4 - \text{gon}, \text{ if } x = e_1.$$
Support functions

The support function of $M \in \mathcal{M}_n$ is the function

$$s_M : \mathbb{R}^n \to \mathbb{R} : s_M(x) = \max_{y \in M} < x, y >,$$

where $< x, y > = \sum_{k=1}^{n} x_k y_k$ is the scalar product.

- $s_M(\lambda x) = \lambda s_M(x)$ for any non negative $\lambda \in \mathbb{R}$. So, if $|x| \neq 0$, then $s_M(x) = |x| s_M\left(\frac{x}{|x|}\right)$.
- $s_M$ is a piece linear function.
- $s_M$ is a linear function iff $M$ is a point in $\mathbb{R}^n$.
- $s_M$ is a convex (concave up) function, that is

$$s_M(tx_1 + (1 - t)x_2) \leq t s_M(x_1) + (1 - t) s_M(x_2)$$

for any $x_1, x_2$ and $t \in [0, 1]$. 
Lemma. For any $M_1, M_2 \in \mathcal{M}$ we have

$$s_{M_1+M_2} = s_{M_1} + s_{M_2}.$$  

Proof. Let $M(x)$ be the image of mapping

$M \rightarrow \mathbb{R} : y \rightarrow <x, y>$. We have $M(x) = [a, b]$, where $a = \max_{y \in M} <x, y> = s_M(x)$ and $b = \min_{y \in M} <x, y>$. It is clear that

$$M_1(x) + M_2(x) = (M_1 + M_2)(x).$$

Using that in $\mathbb{R}^1$

$$[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2],$$

we obtain the proof.
Fan of a convex polytope

Consider the set \( \{ v_1, \ldots, v_N \} \) of all vertex of \( M \). For any \( x \in \mathbb{R}^n \) there exists such \( v_i \) that \( s_M(x) = < x, v_i > \). Set

\[
V_i = \{ x \in \mathbb{R}^n : s_M(x) = < x, v_i > \}.
\]

We have \( \mathbb{R}^n = \bigcup_{i=1}^{N} V_i \) and any \( V_i \) is the convex polyhedral cone with vertex \( O \in \mathbb{R}^n \). The set \( \{ V_i, i = 1, \ldots, N \} \) gives the fan of convex polytope \( M \).

**Example.** \( M = \text{conv}((1, 0), (0, 1)) \subset \mathbb{R}^2 \). Then

\[
V_i = \{ x = (x_1, x_2) \in \mathbb{R}^2 : \max_{y \in M} < x, y > = x_i \},
\]

\[
\max_{y \in M} < x, y > = \max_{t \in [0,1]} \left( tx_1 + (1 - t)x_2 \right).
\]
Minkowski sums in $\mathbb{R}^3$

(Zonohedra)

A zonohedron is a convex polyhedron in $\mathbb{R}^3$ where every face is a polygon with point symmetry or, equivalently, symmetry under rotations through $180^\circ$.

Any zonohedron may equivalently be described as the Minkowski sum of a set of line segments in $\mathbb{R}^3$, or as the 3-dim projection of a hypercube.

Zonohedra were originally defined and studied by E. S. Fedorov (1853–1919), a Russian crystallographer.

More generally, in $\mathbb{R}^n$ the Minkowski sum of line segments forms a polytope known as a zonotope.

The well known rhombic dodecahedron is a convex polyhedron with 12 rhombic faces, 24 edges and 14 vertices. Some minerals such as garnet form a rhombic dodecahedral crystal habit. Honeybees use the geometry of rhombic dodecahedra to form honeycomb.

It gives an example when a Minkowski sum of the simple polytopes forms a nonsimple polytope.
Let $\Delta^1 = I^1 = \{(t_1, t_2), \ t_i \geq 0, \ t_1 + t_2 = 1\}$.
Consider a collection $\{(v_{1,1}, v_{1,2}), \ldots, (v_{N,1}, v_{N,2})\}$
of pairs of vectors in $\mathbb{R}^n$ and the mapping

$$\varphi: I^N \longrightarrow \mathbb{R}^n : \varphi\left((t_{1,1}, t_{1,2}), \ldots, (t_{N,1}, t_{N,2})\right) = \sum_{i=1}^{N} (t_{i,1}v_{i,1} + t_{i,2}v_{i,2}).$$

Image of $\varphi$ is called a zonotope in $\mathbb{R}^n$.

**Problem.** When $\varphi$ gives a simple polytope?

**More general problem.**
Consider a collection

$$\{(v_{1,1}, \ldots, v_{1,i_1+1}), \ldots, (v_{N,1}, \ldots, v_{N,i_N+1})\}$$
of the finite sets of vectors in $\mathbb{R}^n$ and the mapping

$$\Phi: \Delta^{i_1} \times \cdots \times \Delta^{i_N} \longrightarrow \mathbb{R}^n : \Phi(t_1, \ldots, t_N) = \sum_{k=1}^{N} \sum_{j=1}^{i_k+1} t_{k,j}v_{k,j},$$

where $t_k = (t_{k,1}, \ldots, t_{k,i_k+1})$.

**Problem.** When $\Phi$ gives a simple polytope?
Consider a collection $B$ of non-empty subsets of the set $[n] = \{1, \ldots, n\}$. Let $e_i$, $i = 1, \ldots, n$, are the endpoints of the standard basis vectors in $\mathbb{R}^n$.

For any $I \in B$ set $\Delta_I = \text{ConvexHull}\{e_i | i \in I\}$ and $P_B = \sum_{I \in B} \Delta_I$.

Convex polytope $P_B$ is the image of the map $\varphi_B: \prod_{I \in B} \Delta_I \longrightarrow \mathbb{R}^n$.

**Problem.** When $\varphi_B$ gives a simple polytope?

**Definition.** A collection $B$ of non-empty subsets of the set $[n] = \{1, \ldots, n\}$ is called a building set if:
- $I, J \in B$ and $I \cap J \neq \emptyset \Rightarrow I \cup J \in B$
- $\{i\} \in B$ for all $i \in [n]$.

**Theorem.** The convex polytope $P_B \subset \mathbb{R}^n$ is a simple polytope.
**Definition.** Let $\Gamma$ be a graph with vertex set $[n] = \{1, \ldots, n\}$ and no loops or multiple edges. The graphical building set $B(\Gamma)$ is the set of all non-empty subsets $I \subset [n]$ such that the graph $\Gamma|_I$ is connected.

For example for the graph $\Gamma$

```
1 2 3
\_\_\_
|
```

we have

```
1 2 3
```

```
\_\_\_
| 4
```

```
\_\_\_
| 4
```

```
\_\_\_
| 4
```

that is $\{2, 3, 4\} \in B(\Gamma), \quad \text{and} \quad \{1, 3, 4\} \notin B(\Gamma)$.

**Lemma.** The graphical building set $B(\Gamma)$ is a building set.
**Graph-associahedra.**

Given a finite graph $\Gamma$. The graph-associahedron $P(\Gamma)$ is a simple polytope whose face poset is based on the connected subgraph of $\Gamma$. When $\Gamma$ is:

- a path
  
  ![Path Graph](image)

- a cycle
  
  ![Cycle Graph](image)

- a complete graph
  
  ![Complete Graph](image)

- an $n$-star graph
  
  ![Star Graph](image)

the polytope $P(\Gamma)$ results in the:

- associahedron (Stasheff polytope) $As^n$,
- cyclohedron (Bott–Taubes polytope) $Cy^n$,
- permutohedron $Pe^n$,
- stellohedron $St^n$, respectively.

$As^2 = St^2$ is 5 gon and $Cy^2 = Pe^2$ is 6 gon.
GRAPH-ASSOCIAHEDRON

Associahedron $A_{3}^3$

The Stasheff polytope $K_5$. 
GRAPH-ASSOCIAHEDRON

Cyclohedron $C_y^3$

Bott–Taubes polytope
GRAPH-ASSOCIAHEDRON

Permutohedron $P e^3$. 
Theorem. (Andrew G.Fenn) For a connected graph $\Gamma$ on $n + 1$ nodes, we have

$$dP(\Gamma) = \sum_{G} P(\Gamma_G) \times P(\overline{\Gamma}_G^c)$$

where

1. $G \subseteq \{1, \ldots, n + 1\}$

2. $\Gamma_G$ is the subgraph of $\Gamma$ with vertex set $G$.

3. $\overline{\Gamma}_G^c$ is the graph with vertex set $\{1, \ldots, n + 1\}\setminus G$ and arcs between two vertices, $i$ and $j$, if they are path connected in $\Gamma_{G\cup\{i,j\}}$.

4. $G$ runs over all proper subsets of $\{1, \ldots, n + 1\}$ such that $\Gamma_G$ is connected.
We have these formulas for $d(P(\Gamma))$: 

\[
dAs^n = \sum_{i+j=n-1} (i + 2)A_s^i \times As^j
\]

\[
dCy^n = (n + 1) \sum_{i+j=n-1} As^i \times Cy^j
\]

\[
dPe^n = \sum_{i+j=n-1} \left(\frac{n+1}{i+1}\right) Pe^i \times Pe^j
\]

\[
dSt^n = n \cdot St^{n-1} + \sum_{i=0}^{n-1} \binom{n}{i} St^i \times Pe^{n-i-1}
\]
Lemma. Let \([n] \in B\). Then for any \(I_k \in B\) such that \(I_k \neq [n]\), the equation

\[
\sum_{i \in I_k} x_i = \mu(I_k) \tag{\ast}
\]

gives a facet of \(P_B\), where \(\mu(I_k)\) is the number of \(I_l \in B\) such that \(I_l \subset I_k\).

Any facet of \(P_B\) can be described by equation (\ast) for some \(I_k \in B\).

Examples.

\[
\Gamma : \quad \begin{array}{c}
1 \\
2 \\
3
\end{array} \quad P(\Gamma) = As^2 - 5\text{-gon}
\]

\[B = \{ \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\} \}\]

\[
\Gamma : \quad \begin{array}{c}
1 \\
2 \\
3
\end{array} \quad P(\Gamma) = Cy^2 - 6\text{-gon}
\]

\[B = \{ \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\} \}\]