The fundamental groupoid scheme and applications

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The aim of the talk is to report some results concerning the fundamental group scheme of a scheme.
Grothendieck’s arithmetic fundamental group

- $X/k$ finite type, connected, $k$ perfect, $\bar{x} \to X$ geom. point, $\rightsquigarrow$
  $k \subset \bar{k} \subset \kappa(\bar{x})$
Grothendieck’s arithmetic fundamental group

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- Category $\text{ECov}(X)$:
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  Obj: $\pi : Y \rightarrow X$ étale covering
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- Category $\text{ECov}(X)$:
  - Obj: $\pi : Y \rightarrow X$ étale covering
  - Mor: $Y_1 \xrightarrow{h} Y_2$

Fiber functor: $\text{ECov}(X) \xrightarrow{\omega} \text{FSets}$, $\pi \mapsto \pi^{-1}(\bar{x})$

**Defn:** $\pi_1(X, \bar{x}) := \text{Aut}(\omega_{\bar{x}})$

Arithmetic fundamental group, it is a pro-finite group
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  Mor: $\begin{tikzcd}
  Y_1 \arrow[r, h] \arrow[dr, \pi_1] & Y_2 \arrow[dl, \pi_2] \\
  & X
  \end{tikzcd}$

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- **Defn:** $\pi_1(X, \bar{x}) := \text{Aut}(\omega_{\bar{x}})$, *arithmetic fundamental group*, it is a pro-finite group
Grothendieck’s theorem

**Thm:**

- $\operatorname{ECov}(X) \cong \pi_1(X, \bar{x}) \to \pi_1(\bar{X})$ extends to pro-objects $\pi_1(X, \bar{x})$ acting on itself via translations $\sim \tilde{\pi}_{\bar{x}}: \tilde{X}_{\bar{x}} \to X_{\bar{x}}$ (universal covering with base point $\bar{x}$) with $\omega_{\bar{x}}(\tilde{\pi}_{\bar{x}}) = \pi_1(X, \bar{x})$.

- $X$ geometrically connected, $\operatorname{ECov}(k) \subset \operatorname{ECov}(X)$ via pullback.

**Conjecture (Grothendieck):**

Assume $X/k$ hyperbolic curve, $k/\mathbb{Q}$ of finite type. Let $Y \supset X$ be smooth compactification. Then

- (a) sections of $\epsilon$ are determined by $k$-points of $Y$;
- (b) to a $k$-point in $X$ there corresponds a unique section;
- (c) to a $k$-point in $Y \setminus X$ there corresponds a packet of uncountably many sections.
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- **Thm:**
  - Equivalence of categories $\text{ECoV}(X) \xrightarrow{\omega_{\bar{x}}} \pi_1(X, \bar{x}) \to \text{FSets}$
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- $X$ geometrically connected, $\text{ECov}(k) \subset \text{ECov}(X)$ via pullback $\sim$ exact sequence

\[
1 \to \pi_1(\tilde{X}, \bar{x}) \to \pi_1(X, \bar{x}) \xrightarrow{\epsilon} \text{Gal}(\bar{k}) \to 1
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Nori’s fundamental group scheme $\pi^N(X, x)$

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  - finite bundle: satisfies polynomial equation with integral coefficients (in the sense of Grothendieck ring) $\implies$ semistable of degree 0
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- Fiber functor: fixing a $k$-point on $X$ and taking fiber at that point $\sim\pi^N(X, x)$ - pro-finite $k$-group scheme
Comparison with Grothendieck’s group
(Esnault-Hai-Sun, 2007)

- $\text{Char}(k) = 0$: Nori’s fundamental group gives back Grothendieck’s geometric fundamental group $\pi^N(X, x)(\bar{k}) = \pi_1(\bar{X}, \bar{x})$
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- \text{Char}(k) = p > 0: \text{the pro-étale quotient of } \pi^N(X, x) \text{ gives back } \pi_1(\bar{X}, \bar{x}).
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- **Char($k$)$=p > 0$**: the pro-étale quotient of $\pi^N(X, x)$ gives back $\pi_1(\bar{X}, \bar{x})$.
- **Case of finite group scheme $H$**

The composition $H_{\text{red}} \rightarrow H \rightarrow H_{\text{et}}$ is an isomorphism $\Rightarrow H = H_0 \ltimes H_{\text{et}}$

The composition $H_0 \rightarrow H \rightarrow H_{\text{loc}}$ is not necessarily an isomorphism ($H_{\text{loc}}$ is the largest local quotient of $H$).

For $\pi_N(X, x)$ (pro-finite group scheme) Tannaka duality defines the quotient maps $\pi_N \rightarrow \pi_F$ – the pro-local quotient, $\pi_N \rightarrow \pi_{\text{ét}}$ – the pro-étale quotient.

$\pi_N^0 = \ker(\pi_N \rightarrow \pi_{\text{ét}})$ the connected component of 1 $\pi_N^0$ may be larger than $\pi_F(X, x)$ one has a splitting $\pi_{\text{ét}} \rightarrow \pi_N \Rightarrow \pi_N$ is a semi-direct product of $\pi_{\text{ét}}$ and $\pi_N^0$.

If $\pi_N$ is commutative then the above product is direct.

**Question**: is the converse true?
Comparison with Grothendieck’s group
(Esnault-Hai-Sun, 2007)

- Char($k$) = 0: Nori’s fundamental group gives back Grothendieck’s geometric fundamental group $\pi^N(X, x)(\overline{k}) = \pi_1(\overline{X}, \overline{x})$
- Char($k$) = $p > 0$: the pro-étale quotient of $\pi^N(X, x)$ gives back $\pi_1(\overline{X}, \overline{x})$.
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- \textbf{Char}(k)\equiv 0: \text{Nori’s fundamental group gives back Grothendieck’s geometric fundamental group } \pi^N(X, x)(\bar{k}) = \pi_1(\bar{X}, \bar{x})
- \text{Char}(k)\equiv p > 0: \text{the pro-étale quotient of } \pi^N(X, x) \text{ gives back } \pi_1(\bar{X}, \bar{x})

\textbf{Case of finite group scheme } H

- The composition \( H_{\text{red}} \to H \to H^{\text{ét}} \) is an isomorphism \( \sim \to H = H^0 \ltimes H^{\text{ét}} \)
- The composition \( H^0 \to H \to H^{\text{loc}} \) is not necessarily an isomorphism (\( H^{\text{loc}} \) is the largest local quotient of \( H \)).
Comparison with Grothendieck’s group
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- **Char**($k$)$=0$: Nori’s fundamental group gives back Grothendieck’s geometric fundamental group $\pi^N(X, x)(\bar{k}) = \pi_1(\bar{X}, \bar{x})$
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**For** $\pi^N(X, x)$ (pro-finite group scheme)
Comparison with Grothendieck’s group
(Esnault-Hai-Sun, 2007)

- **Char\(k\) = 0**: Nori’s fundamental group gives back Grothendieck’s geometric fundamental group \(\pi^N(X, x)(\overline{k}) = \pi_1(\overline{X}, \overline{x})\)
- **Char\(k\) = \(p > 0\)**: the pro-étale quotient of \(\pi^N(X, x)\) gives back \(\pi_1(\overline{X}, \overline{x})\).

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**For \(\pi^N(X, x)\) (pro-finite group scheme)**
- Tannaka duality defines the quotient maps \(\pi^N \rightarrow \pi^F\) – the pro-local quotient, \(\pi^N \rightarrow \pi^{\text{ét}}\) – the pro-étale quotient.
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- \( \text{Char}(k) = 0 \): Nori’s fundamental group gives back Grothendieck’s geometric fundamental group \( \pi^N(X, x)(\bar{k}) = \pi_1(\bar{X}, \bar{x}) \)
- \( \text{Char}(k) = p > 0 \): the pro-étale quotient of \( \pi^N(X, x) \) gives back \( \pi_1(\bar{X}, \bar{x}) \).

**Case of finite group scheme** \( H \)

- The composition \( H_{\text{red}} \to H \to \text{H ét} \) is an isomorphism \( \cong H = H^0 \times \text{H ét} \)
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- Tannaka duality defines the quotient maps \( \pi^N \to \pi^F \) – the pro-local quotient, \( \pi^N \to \pi^\text{ét} \) – the pro-étale quotient.
- \( \pi^N_0 = \text{Ker}(\pi^N \to \pi^\text{ét}) \) the connected component of 1
Comparison with Grothendieck’s group
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- \(\text{Char}(k)=0\) : Nori’s fundamental group gives back Grothendieck’s geometric fundamental group \(\pi^N(X, x)(\bar{k}) = \pi_1(\bar{X}, \bar{x})\)
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- \(\pi^N_0\) may be larger than \(\pi^F(X, x)\)
Comparison with Grothendieck’s group  
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- \( \text{Char}(k)=0 \) : Nori’s fundamental group gives back Grothendieck’s geometric fundamental group \( \pi^N(X, x)(\overline{k}) = \pi_1(\overline{X}, \overline{x}) \)
- \( \text{Char}(k)=p > 0 \) : the pro-étale quotient of \( \pi^N(X, x) \) gives back \( \pi_1(\overline{X}, \overline{x}) \).

**Case of finite group scheme** \( H \)
- The composition \( H_{\text{red}} \to H \to H^\text{ét} \) is an isomorphism \( \hookrightarrow H = H^0 \ltimes H^\text{ét} \)
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- \( \pi^N_0 = \text{Ker}(\pi^N \to \pi^\text{ét}) \) the connected component of 1
- \( \pi^N_0 \) may be larger than \( \pi^F(X, x) \)
- one has a splitting \( \pi^\text{ét} \to \pi^N \)
Comparison with Grothendieck’s group
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- Char$(k)=0$ : Nori’s fundamental group gives back Grothendieck’s geometric fundamental group $\pi^N(X, x)(\bar{k}) = \pi_1(\bar{X}, \bar{x})$
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Case of finite group scheme $H$
- The composition $H_{\text{red}} \to H \to H^{\text{ét}}$ is an isomorphism $\leadsto H = H^0 \rtimes H^{\text{ét}}$
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- Tannaka duality defines the quotient maps $\pi^N \to \pi^F$ – the pro-local quotient, $\pi^N \to \pi^{\text{ét}}$ – the pro-étale quotient.
- $\pi^N_0 = \text{Ker}(\pi^N \to \pi^{\text{ét}})$ the connected component of 1
- $\pi^N_0$ may be larger than $\pi^F(X, x)$
- one has a splitting $\pi^{\text{ét}} \to \pi^N$
- $\leadsto \pi^N$ is a semi-direct product of $\pi^{\text{ét}}$ and $\pi^N_0$
Comparison with Grothendieck’s group
(Esnault-Hai-Sun, 2007)

- **Char**(\(k\))\(=0\) : Nori’s fundamental group gives back Grothendieck’s geometric fundamental group \(\pi^N(X, x)(k) = \pi_1(\bar{X}, \bar{x})\)

- **Char**(\(k\))\(=p > 0\): the pro-étale quotient of \(\pi^N(X, x)\) gives back \(\pi_1(\bar{X}, \bar{x})\).

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- \(\pi^N_0 = \text{Ker}(\pi^N \to \pi^{\text{ét}})\) the connected component of 1
- \(\pi^N_0\) may be larger than \(\pi^F(X, x)\)
- one has a splitting \(\pi^{\text{ét}} \to \pi^N\)
  \(\sim \) \(\pi^N\) is a semi-direct product of \(\pi^{\text{ét}}\) and \(\pi^N_0\)
- If \(\pi^N\) is commutative then the a above product is direct
Comparison with Grothendieck’s group
(Esnault-Hai-Sun, 2007)

- \( \text{Char}(k)=0 \): Nori’s fundamental group gives back Grothendieck’s geometric fundamental group \( \pi^N(X, x)(\overline{k}) = \pi_1(\overline{X}, \overline{x}) \)
- \( \text{Char}(k)=p > 0 \): the pro-étale quotient of \( \pi^N(X, x) \) gives back \( \pi_1(\overline{X}, \overline{x}) \).

Case of finite group scheme \( H \)
- The composition \( H_{\text{red}} \to H \to H^\text{ét} \) is an isomorphism \( \leadsto H = H^0 \rtimes H^\text{ét} \)
- The composition \( H^0 \to H \to H^\text{loc} \) is not necessarily an isomorphism (\( H^\text{loc} \) is the largest local quotient of \( H \)).

For \( \pi^N(X, x) \) (pro-finite group scheme)
- Tannaka duality defines the quotient maps \( \pi^N \to \pi^F \) – the pro-local quotient, \( \pi^N \to \pi^\text{ét} \) – the pro-étale quotient.
- \( \pi^N_0 = \ker(\pi^N \to \pi^\text{ét}) \) the connected component of 1
- \( \pi^N_0 \) may be larger than \( \pi^F(X, x) \)
- one has a splitting \( \pi^\text{ét} \to \pi^N \)
- \( \leadsto \pi^N \) is a semi-direct product of \( \pi^\text{ét} \) and \( \pi^N_0 \)
- If \( \pi^N \) is commutative then the above product is direct
- Question: is the converse true?
Case $X/k$ smooth, char $k = 0$

The fundamental groupoid schemes $\Pi(X, \bar{x})$ (Esnault-Hai, 2008)

- **Problem**: give a notion of fundamental group scheme for a not necessarily complete scheme.
Case $X/k$ smooth, char $k = 0$

The fundamental groupoid schemes $\Pi(X, \bar{x})$ (Esnault-Hai, 2008)

- **Problem**: give a notion of fundamental group scheme for a not necessarily complete scheme.
- **Idea**: apply general Tannaka duality to the tensor category of finite connections $\rightsquigarrow$ fundamental groupoid scheme $\Pi(X, \bar{x})$
Case $X/k$ smooth, char $k = 0$

The fundamental groupoid schemes $\Pi(X, \bar{x})$ (Esnault-Hai, 2008)

- **Problem**: give a notion of fundamental group scheme for a not necessarily complete scheme.

- **Idea**: apply general Tannaka duality to the tensor category of finite connections $\leadsto$ fundamental groupoid scheme $\Pi(X, \bar{x})$

- **Finite connection**: a connection satisfying a polynomial equation with coefficients from $\mathbb{Z}$
Case $X/k$ smooth, char $k = 0$

The fundamental groupoid schemes $\Pi(X, \bar{x})$ (Esnault-Hai, 2008)

- **Problem**: give a notion of fundamental group scheme for a not necessarily complete scheme.

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- Finite connection: a connection satisfying a polynomial equation with coefficients from $\mathbb{Z}$

- **General Tannaka duality**:

  $$\left(\begin{array}{l}
  \text{Tensor category over } k + \\
  \text{fiber functor to } \text{Vect}_K, K \supset k
  \end{array}\right) \overset{1-1}{\leftrightarrow} \left(\begin{array}{l}
  \text{Group scheme over } k \\
  \text{acting transitively on } K
  \end{array}\right)$$
Case \( X/k \) smooth, char\( k = 0 \)

The fundamental groupoid schemes \( \Pi(X, \bar{x}) \) (Esnault-Hai, 2008)

- **Problem**: give a notion of fundamental group scheme for a not necessarily complete scheme.
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\[
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\begin{pmatrix}
\text{Group scheme over } k \\
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\end{pmatrix}
\]

- **Why general Tannaka duality (fiber functor to } \text{Vect}_K \rangle\): to free ourselves from the existence of \( k \)-rational points.
**Case $X/k$ smooth, char$k = 0$**

**The fundamental groupoid schemes** $\Pi(X, \bar{x})$ *(Esnault-Hai, 2008)*

- **Problem**: give a notion of fundamental group scheme for a not necessarily complete scheme.
- **Idea**: apply general Tannaka duality to the tensor category of finite connections $\leadsto$ fundamental groupoid scheme $\Pi(X, \bar{x})$.
- **Finite connection**: a connection satisfying a polynomial equation with coefficients from $\mathbb{Z}$.
- **General Tannaka duality**:

$$
\begin{array}{c}
\left( \begin{array}{c}
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\text{Group scheme over } k \\
\text{acting transitively on } K
\end{array} \right)
\end{array}
$$

- Why general Tannaka duality (fiber functor to $\text{Vect}_K$): to free ourselves from the existence of $k$-rational points.
- **Advantage**: $\Pi(X, \bar{x})$ gives back the **arithmetic** fundamental group.
Case $X/k$ smooth, $\text{char} k = 0$

Recall: Affine groupoid schemes

- Affine groupoid scheme $\Pi/k$ acting on $S := \text{Spec}(K)/k$:
  
  $(t, s) : \Pi \to S \times_k S$ affine morphism such that:
  
  - $\Pi$ acts transitively on $S$ if for any $a, b \in \text{Obj}(\Pi)$ there exists $u : T' \to T$ faithfully flat, such that $\text{Mor}(a \circ u, b \circ u) \neq \emptyset$
  
  $\leftrightarrow (t, s)$ is a faithfully flat morphism
Case $X/k$ smooth, $\text{char} k = 0$
Recall: Affine groupoid schemes

Affine groupoid scheme $\Pi/k$ acting on $S := \text{Spec}(K)/k$:
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- for any $k$-scheme $T$, one has a groupoid $\Pi(T)$
Case $X/k$ smooth, $\text{char} k = 0$

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Case $X/k$ smooth, $\text{ch} k = 0$

Recall: Affine groupoid schemes

- Affine groupoid scheme $\Pi/k$ acting on $S := \text{Spec}(K)/k$: $(t, s) : \Pi \to S \times_k S$ affine morphism such that:
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Case $X/k$ smooth, char $k = 0$

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Case $X/k$ smooth, $\text{chark} = 0$

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Case $X/k$ smooth, $\text{char} k = 0$

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- Affine groupoid scheme $\Pi/k$ acting on $S := \text{Spec}(K)/k$:
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  for any $a, b \in \text{Obj} \Pi(T)$ there exists $u : T' \to T$ faithfully flat, such that $\text{Mor}(a \circ u, b \circ u) \neq \emptyset$
    $\iff (t, s)$ is a faithfully flat morphism
Case $X/k$ smooth, char $k = 0$

Recall: Affine groupoid schemes

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    that $\text{Mor}(a \circ u, b \circ u) \neq \emptyset$
    $(t, s)$ is a faithfully flat morphism

- The diagonal group scheme: $\Delta : S \to S \times_k S$ the diagonal map
Case $X/k$ smooth, $\text{char} k = 0$

Recall: Affine groupoid schemes

- Affine groupoid scheme $\Pi/k$ acting on $S := \text{Spec}(K)/k$: $(t, s) : \Pi \to S \times_k S$ affine morphism such that:
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    - $(t, s)$ is a faithfully flat morphism

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  - $\Pi^\Delta := \Pi \times_{S \times S} S$, $\Pi^\Delta$ is a flat $S$-group scheme,
Case $X/k$ smooth, $\text{char} k = 0$

Recall: Affine groupoid schemes

- Affine groupoid scheme $\Pi/k$ acting on $S := \text{Spec}(K)/k$: $(t, s) : \Pi \to S \times_k S$ affine morphism such that:
  - for any $k$-scheme $T$, one has a groupoid $\Pi(T)$
    - $\text{Obj}(\Pi(T)) := \text{Mor}_k(T, S)$ (i.e. $T$-points of $S$)
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  - $\Pi$ acts transitively on $S$ if
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    $(t, s)$ is a faithfully flat morphism

- The diagonal group scheme: $\Delta : S \to S \times_k S$ the diagonal map
  - $\Pi^\Delta := \Pi \times_{S \times S} S$, $\Pi^\Delta$ is a flat $S$-group scheme,
  - exact sequence of groupoid schemes $\Pi^\Delta \to \Pi \to S \times_k S$
    $\downarrow$ $\downarrow$ $\downarrow$
    $S \xrightarrow{\Delta} S \times_k S \xrightarrow{=} S \times_k S$

P.H. Hai (Inst of Math-HN)
Case $X/k$ smooth, char $k = 0$

The fundamental groupoid schemes $\Pi(X, \bar{x})$ (Esnault-Hai, 2008)

- **Settings:**

  - $X/k$ smooth, char $k = 0$
  - Settings:
  - $FConn(X)$ (finite connections) is a $k$-linear abelian rigid tensor category
  - $\bar{x}: \bar{k} \to X$ geometric point defines fibre functor to $\text{Vec}_{\bar{k}}$
  - Tannaka duality yields $\Pi(X, \bar{x})$ – the fundamental groupoid scheme based at $\bar{x}$
  - Compatibility with base changes:
  - $\Pi(\bar{X}, \bar{x}) \cong \Pi(X, x) \Delta \rightarrow \Pi(X, \bar{x}) \Delta \rightarrow S \times_k S \Delta \rightarrow S \times_k S=$
Case $X/k$ smooth, $\text{char} k = 0$

The fundamental groupoid schemes $\Pi(X, \bar{x})$ (Esnault-Hai, 2008)

- **Settings:**
  - $X/k$ smooth, geometrically connected, $\text{char}(k) = 0$, 

...
Case $X/k$ smooth, char $k = 0$

The fundamental groupoid schemes $\Pi(X, \bar{x})$ (Esnault-Hai, 2008)

- **Settings:**
  - $X/k$ smooth, geometrically connected, char($k$) = 0,
  - $\text{FConn}(X)$ (finite connections) is a $k$-linear abelian rigid tensor category
Case $X/k$ smooth, $\text{char} k = 0$

The fundamental groupoid schemes $\Pi(X, \bar{x})$ (Esnault-Hai, 2008)

- **Settings:**
  - $X/k$ smooth, geometrically connected, $\text{char}(k) = 0$,  
  - $\text{FConn}(X)$ (finite connections) is a $k$-linear abelian rigid tensor category
  - $\bar{x} : \bar{k} \to X$ geometric point defines fibre functor to $\text{Vec}_{\bar{k}}$
Case $\mathcal{X}/k$ smooth, $\text{char} k = 0$

The fundamental groupoid schemes $\Pi(\mathcal{X}, \bar{x})$ (Esnault-Hai, 2008)

- **Settings:**
  - $\mathcal{X}/k$ smooth, geometrically connected, $\text{char}(k) = 0$,
  - $\text{FConn}(\mathcal{X})$ (finite connections) is a $k$-linear abelian rigid tensor category
  - $\bar{x} : \bar{k} \to \mathcal{X}$ geometric point defines fibre functor to $\text{Vec}_{\bar{k}}$
  - $\Leftrightarrow$ Tannaka duality yields $\Pi(\mathcal{X}, \bar{x})$ – the fundamental groupoid scheme based at $\bar{x}$
Case $X/k$ smooth, char$k = 0$

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- **Settings:**
  - $X/k$ smooth, geometrically connected, char$(k) = 0$,
  - $\text{FConn}(X)$ (finite connections) is a $k$-linear abelian rigid tensor category
  - $\bar{x} : \bar{k} \to X$ geometric point defines fibre functor to $\text{Vec}_{\bar{k}}$
  - $\rightsquigarrow$ Tannaka duality yields $\Pi(X, \bar{x})$ – the fundamental groupoid scheme based at $\bar{x}$

- **Compatibility with base changes:**
Case $X/k$ smooth, char$k = 0$

The fundamental groupoid schemes $\Pi(X, \bar{x})$ (Esnault-Hai, 2008)

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  - $\bar{x}: \bar{k} \to X$ geometric point defines fibre functor to $\text{Vec}_{\bar{k}}$
  - \(\leadsto\) Tannaka duality yields $\Pi(X, \bar{x})$ – the fundamental groupoid scheme based at $\bar{x}$

- **Compatibility with base changes:**
  - $\Pi(\bar{X}, \bar{x}) \cong \Pi(X, x)^\Delta$
Case $X/k$ smooth, char$k = 0$
The fundamental groupoid schemes $\Pi(X, \bar{x})$ (Esnault-Hai, 2008)

- **Settings:**
  - $X/k$ smooth, geometrically connected, char$(k) = 0$,
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  - $\bar{x} : \bar{k} \to X$ geometric point defines fibre functor to $\text{Vec}_{\bar{k}}$
  - Tannaka duality yields $\Pi(X, \bar{x})$ – the fundamental groupoid scheme based at $\bar{x}$

- **Compatibility with base changes:**
  - $\Pi(\bar{X}, \bar{x}) \cong \Pi(X, x)^\Delta$
  - Exact sequence
    \[
    \begin{align*}
    \Pi(\bar{X}, \bar{x}) = \Pi(X, x)^\Delta & \to \Pi(X, \bar{x}) \to S \times_k S \\
    S & \to S \times_k S \cong S \times_k S
    \end{align*}
    \]
Case $X/k$ smooth, $\text{char} k = 0$

Compare with Grothendieck group (Esnault-Hai, 2008)

- Compare with Grothendieck fundamental group

\[
\pi^1(\bar{X}, \bar{x}) \sim \Pi(\bar{k}) = \Delta(\bar{k})
\]

\[
\pi^1(X, \bar{x}) \sim \Pi_s(\bar{k}) \Rightarrow \Delta(\bar{k})
\]

\[
\Rightarrow \text{Commutative diagram of exact lines}
\]

\[
\Rightarrow \Rightarrow \Rightarrow \Rightarrow
\]

Consequence: there is one-one correspondence between neutral fiber functors of $FConn(X)$ and section to $\epsilon$ (up to conjugations by $\pi^1(\bar{X}, \bar{x})$).

\[
\Rightarrow \text{answer questions (b), (c) of section conjecture.}
\]
Case $X/k$ smooth, char $k = 0$

Compare with Grothendieck group (Esnault-Hai, 2008)

- Compare with Grothendieck fundamental group
  - $\pi_1(\bar{X}, \bar{x}) \cong \Pi(X)^A(\bar{k})$
Case $X/k$ smooth, $\text{char} k = 0$

Compare with Grothendieck group (Esnault-Hai, 2008)

- Compare with Grothendieck fundamental group
  - $\pi_1(\bar{X}, \bar{x}) \cong \Pi(X)^{\Delta}(\bar{k})$
  - $\pi_1(X, \bar{x}) \cong \Pi_s(X, \bar{x})(\bar{k})$

Consequence: there is one-one correspondence between neutral fiber functors of $FConn(X)$ and section to $\epsilon$ (up to conjugations by $\pi_1(\bar{X}, \bar{x})$).

Answer questions (b), (c) of section conjecture.
Case $X/k$ smooth, char $k = 0$

Compare with Grothendieck group (Esnault-Hai, 2008)

- Compare with Grothendieck fundamental group
  - $\pi_1(\tilde{X}, \tilde{x}) \cong \Pi(X)^A(\tilde{k})$
  - $\pi_1(X, \bar{x}) \cong \Pi_s(X, \bar{x})(\bar{k})$
- Compatibility with $\text{Gal}(\bar{k}/k)$

Consequence: there is one-one correspondence between neutral fiber functors of $\text{FConn}(X)$ and section to $\epsilon$ (up to conjugations by $\pi_1(\tilde{X}, \tilde{x})$).

$\Rightarrow$ answer questions (b), (c) of section conjecture.
Case $X/k$ smooth, $\text{char} k = 0$

Compare with Grothendieck fundamental group (Esnault-Hai, 2008)

- Compare with Grothendieck fundamental group
  - $\pi_1(\bar{X}, \bar{x}) \cong \Pi(X)^\Delta(\bar{k})$
  - $\pi_1(X, \bar{x}) \cong \Pi_s(X, \bar{x})(\bar{k})$

- Compatibility with $\text{Gal}(\bar{k}/k)$
  - For $\Pi = \text{Spec} \bar{k} \times \text{Spec} \bar{k}$ then $(\text{Spec} \bar{k} \times \text{Spec} \bar{k})(\bar{k}) = \text{Gal}(\bar{k}/k)$
Case $X/k$ smooth, char $k = 0$

Compare with Grothendieck group (Esnault-Hai, 2008)

- Compare with Grothendieck fundamental group
  - $\pi_1(\tilde{X}, \tilde{x}) \cong \Pi(X)^{\Delta}(\bar{k})$
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  - $(t, s) : \Pi \to \text{Spec}\bar{k} \times \text{Spec}\bar{k} \Rightarrow \Pi_s(\bar{k}) \to \text{Gal}(\bar{k}/k)$

\[ \Rightarrow \] Comutative diagram of exact lines
\[ \Rightarrow \] Consequence: there is one-one correspondence between neutral fiber functors of $\text{FConn}(X)$ and section to $\epsilon$ (up to conjugations by $\pi_1(\tilde{X}, \tilde{x})$).\[ \Rightarrow \] answers questions (b), (c) of section conjecture.
Case $X/k$ smooth, $\text{char } k = 0$

Compare with Grothendieck group (Esnault-Hai, 2008)

- Compare with Grothendieck fundamental group
  - $\pi_1(\bar{X}, \bar{x}) \cong \Pi(X)^A(\bar{k})$
  - $\pi_1(X, \bar{x}) \cong \Pi_s(X, \bar{x})(\bar{k})$

- Compatibility with $\text{Gal}(\bar{k}/k)$
  - For $\Pi = \text{Spec}\bar{k} \times \text{Spec}\bar{k}$ then $(\text{Spec}\bar{k} \times \text{Spec}\bar{k})(\bar{k}) = \text{Gal}(\bar{k}/k)$
  - $(t, s): \Pi \rightarrow \text{Spec}\bar{k} \times \text{Spec}\bar{k} \Rightarrow \Pi_s(\bar{k}) \rightarrow \text{Gal}(\bar{k}/k)$

\[\Rightarrow\] Commutative diagram of exact lines

\[
\begin{array}{ccc}
\Pi(\bar{X}, \bar{x})(\bar{k}) & \longrightarrow & \Pi_s(\bar{k}) \\
\downarrow & & \downarrow \\
\pi_1(\bar{X}, \bar{x}) & \longrightarrow & \pi_1(X, \bar{x}) \xrightarrow{\epsilon} \text{Gal}(\bar{k}/k)
\end{array}
\]
Case $X/k$ smooth, char$k = 0$

Compare with Grothendieck group (Esnault-Hai, 2008)

- Compare with Grothendieck fundamental group
  - $\pi_1(\bar{X}, \bar{x}) \cong \Pi(X)^{\Delta}(\bar{k})$
  - $\pi_1(X, \bar{x}) \cong \Pi_s(X, \bar{x})(\bar{k})$

- Compatibility with $\text{Gal}(\bar{k}/k)$
  - For $\Pi = \text{Spec}\bar{k} \times \text{Spec}\bar{k}$ then $(\text{Spec}\bar{k} \times \text{Spec}\bar{k})(\bar{k}) = \text{Gal}(\bar{k}/k)$
  - $(t, s) : \Pi \to \text{Spec}\bar{k} \times \text{Spec}\bar{k} \Rightarrow \Pi_s(\bar{k}) \to \text{Gal}(\bar{k}/k)$

Commutative diagram of exact lines

\[
\begin{array}{ccc}
\Pi(\bar{X}, \bar{x})(\bar{k}) & \longrightarrow & \Pi_s(\bar{k}) \\
\downarrow & = & \downarrow \\
\pi_1(\bar{X}, \bar{x}) & \longrightarrow & \pi_1(X, \bar{x}) \rightarrow^\epsilon \text{Gal}(\bar{k}/k)
\end{array}
\]

- Consequence:
Case $X/k$ smooth, char $k = 0$

Compare with Grothendieck group (Esnault-Hai, 2008)

- Compare with Grothendieck fundamental group
  - $\pi_1(\bar{X}, \bar{x}) \cong \Pi(X)^{\Delta}(\bar{k})$
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  - For $\Pi = \text{Spec}\bar{k} \times \text{Spec}\bar{k}$ then $(\text{Spec}\bar{k} \times \text{Spec}\bar{k})(\bar{k}) = \text{Gal}(\bar{k}/k)$
  - $(t, s) : \Pi \to \text{Spec}\bar{k} \times \text{Spec}\bar{k} \Rightarrow \Pi_s(\bar{k}) \to \text{Gal}(\bar{k}/k)$

\[ \begin{array}{ccc}
\Pi(\bar{X}, \bar{x})(\bar{k}) & \longrightarrow & \Pi_s(\bar{k}) \\
\longdownarrow & & \longdownarrow \\
\pi_1(\bar{X}, \bar{x}) & \longrightarrow & \pi_1(X, \bar{x}) \\
\epsilon & \longrightarrow & \text{Gal}(\bar{k}/k)
\end{array} \]

\[ \cong \]

- Commutative diagram of exact lines

- Consequence:
  - there is one-one correspondence between neutral fiber functors of $\text{FConn}(X)$ and section to $\epsilon$ (up to conjugations by $\pi_1(\bar{X}, \bar{x})$).
Case $X/k$ smooth, char$k = 0$

Compare with Grothendieck group (Esnault-Hai, 2008)

- Compare with Grothendieck fundamental group
  - $\pi_1(\bar{X}, \bar{x}) \cong \Pi(X)^{\Delta}(\bar{k})$
  - $\pi_1(X, \bar{x}) \cong \Pi_s(X, \bar{x})(\bar{k})$

- Compatibility with $\text{Gal}(\bar{k}/k)$
  - For $\Pi = \text{Spec}\bar{k} \times \text{Spec}\bar{k}$ then $(\text{Spec}\bar{k} \times \text{Spec}\bar{k})(\bar{k}) = \text{Gal}(\bar{k}/k)$
  - $(t, s): \Pi \to \text{Spec}\bar{k} \times \text{Spec}\bar{k} \Rightarrow \Pi_s(\bar{k}) \to \text{Gal}(\bar{k}/k)$

$\leadsto$ Commutative diagram of exact lines

\[
\begin{array}{ccc}
\Pi(\bar{X}, \bar{x})(\bar{k}) & \longrightarrow & \Pi_s(\bar{k}) \\
\downarrow & & \downarrow \\
\pi_1(\bar{X}, \bar{x}) & \longrightarrow & \pi_1(X, \bar{x}) \rightarrow \epsilon \rightarrow \text{Gal}(\bar{k}/k)
\end{array}
\]

- Consequence:
  - there is one-one correspondence between neutral fiber functors of $\text{FConn}(X)$ and section to $\epsilon$ (up to conjugations by $\pi_1(\bar{X}, \bar{x})$).
  - $\leadsto$ answer questions (b), (c) of section conjecture.
Case $X/k$ smooth, $\text{char}(k) = p > 0$, $k = \bar{k}$

$\Pi^{\text{Str}}$ (Dos Santos)

- Idea: apply Tannaka duality to stratified (flat) bundles
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- $\Pi^{\text{Str}}(X, x)$: apply Tannaka duality to $\text{Str}(X)$ and fiber functor at a point $x \in X(k)$. 
Case $X/k$ smooth, $\text{char}(k) = p > 0$, $k = \bar{k}$

$\Pi^\text{Str}$ (Dos Santos)

- Properties:
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$\Pi^{\text{Str}}$ (Dos Santos)

**Properties:**
- $\Pi^{\text{Str}}(X, x)$ is a smooth (pro-algebraic) group scheme
Case $X/k$ smooth, char$(k) = p > 0$, $k = \bar{k}$

$\Pi^{Str}$ (Dos Santos)

**Properties:**

- $\Pi^{Str}(X, x)$ is a smooth (pro-algebraic) group scheme
- The pro-finite quotient $\Pi^{FStr}(X, x)$ is pro-étale
Case $X/k$ smooth, $\text{char}(k) = p > 0$, $k = \overline{k}$

$\Pi_{\text{Str}}$ (Dos Santos)

- **Properties:**
  - $\Pi_{\text{Str}}(X, x)$ is a smooth (pro-algebraic) group scheme
  - The pro-finite quotient $\Pi_{F\text{Str}}(X, x)$ is pro-étale
  - Assuming $X$ is complete, then $\Pi_{\text{Str}}(X, x)$ has no unipotent quotients, on the other hand $\Pi_{F\text{Str}}$ coincides with the pro-étale quotient of Nori’s fundamental group

Questtion: describe the local part? (Gieseker) does $\Pi_{F\text{Str}} = 1$ imply $\Pi_{\text{Str}} = 1$ toward an answer (Dos Santos, Esnault-Hai): $\Pi_{F\text{Str}} = 1$ implies $\Pi_{\text{Str}}$ has no solvable quotients
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