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Lecture II

*Manifolds as the fixed point sets*

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Let $G$ be a compact Lie group acting smoothly on a smooth manifold $M$. Then the set

$$F = \{ x \in M : gx = x \text{ for all } g \in G \}$$

of points fixed under the action of $G$ is a smooth submanifold of $M$ such that $\partial M \cap F = \partial F$.

In particular, if $\partial M = \emptyset$, then $\partial F = \emptyset$.

Moreover, if $M$ is compact, so is $F$.

**Basic question.** Given compact Lie group $G$, which smooth manifolds $F$ occur as the fixed point sets of smooth action of $G$ on smooth manifolds $M$ with prescribed properties?

We focus on the situation $M$ is contractible, and in particular, we deal with the cases where $M$ is a disk or Euclidean space.
Henceforth, we say that a smooth manifold $F$ is \textit{stably complex} if the tangent bundle $\tau_F$ of $F$ admits a complex structure, possibly, after adding to $\tau_F$ a product bundle $F \times \mathbb{R}^n$, which amounts to saying that $F$ admits a smooth embedding into some Euclidean space and the normal bundle admits a complex structure.

\textbf{Theorem A.} Let $G$ be a finite $p$-group or its extension by a torus. Then a smooth manifold $F$ is diffeomorphic to $M^G$ for a smooth action of $G$ on a disk (resp. Euclidean space) $M$ if and only if $F$ is compact (resp. $\partial F = \emptyset$) and $F$ is $\mathbb{Z}_p$-acyclic and stably complex.

\textbf{Theorem B.} Let $G$ be a torus. Then a smooth manifold $F$ is diffeomorphic to $M^G$ for a smooth action of $G$ on a disk (resp. Euclidean space) $M$ if and only if $F$ is $\mathbb{Z}$-acyclic.
Consider six mutually disjoint classes of finite groups $G$ by assuming that $G$ is not of prime power order and the following holds:

$\mathcal{A}$ : $G$ has a $pq$-dihedral subquotient.

$\mathcal{B}$ : $G \not\in \mathcal{A}$ and $G$ has a $pq$-element $g \sim g^{-1}$.

$\mathcal{C}$ : $G$ has a $pq$-element $g \not\sim g^{-1}$ and $G_2 \ntriangleleft G$.

$\mathcal{D}$ : $G$ has a $pq$-element $g \not\sim g^{-1}$ and $G_2 \triangleleft G$.

$\mathcal{E}$ : $G$ has no $pq$-element and $G_2 \ntriangleleft G$.

$\mathcal{F}$ : $G$ has no $pq$-element and $G_2 \triangleleft G$. 
The complexification of real bundles:

\[ c_{\mathbb{R}} : \widetilde{KO}(F) \rightarrow \widetilde{K}(F) \]

The quaternionization of complex bundles:

\[ q_{\mathbb{C}} : \widetilde{K}(F) \rightarrow \widetilde{KSp}(F) \]

The complexification of symplectic bundles:

\[ c_{\mathbb{H}} : \widetilde{KSp}(F) \rightarrow \widetilde{K}(F) \]

The realification of complex bundles:

\[ r_{\mathbb{C}} : \widetilde{K}(F) \rightarrow \widetilde{KO}(F) \]

For an abelian group \( A \), let \( \text{qdiv} A \) denote the intersection of the kernels \( \text{Ker} f \) of all homomorphisms \( f \) from \( A \) to free abelian groups. If \( A \) is finitely generated, then \( \text{qdiv} A \) is the subgroup of torsion elements of \( A \).
Theorem C. Let $G$ be a finite group not of prime power order. Then a smooth manifold $F$ is diffeomorphic to $M^G$ for a smooth action of $G$ on a disk (resp. Euclidean space) $M$ if and only if $F$ is compact and $\chi(M) \equiv 1 \pmod{n_G}$ (resp. $\partial F = \emptyset$) and the following holds:

$G \in \mathcal{A} :$ there is no restriction on $\tau_F$ in $\widetilde{KO}(F)$.

$G \in \mathcal{B} : c_\mathbb{R}(\tau_F) \in c_\mathbb{H}(\widetilde{KSp}(F)) + q\text{div} \, \widetilde{K}(F)$

$G \in \mathcal{C} : \tau_F \in r_\mathbb{C}(\widetilde{K}(F)) + q\text{div} \, \widetilde{KO}(F)$

$G \in \mathcal{D} : \tau_F \in r_\mathbb{R}(\widetilde{K}(F))$

$G \in \mathcal{E} : \tau_F \in q\text{div} \, \widetilde{KO}(F)$

$G \in \mathcal{F} : \tau_F \in r_\mathbb{C}(q\text{div} \, \widetilde{K}(F))$. 