An Optimal Boundedness for Weak $\mathbb{Q}$-Fano Threefolds

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November 11, 2008
Let $V$ be a nonsingular projective variety. Denote by $K_V$ a canonical divisor of $V$. One of the main strategies of explicit birational geometry is to find a practical constant $c = c(V)$ such that the pluricanonical map $\varphi_c$ or $\varphi_{-c}$ has the stable behavior (namely, being the Iitaka fibration).
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♦ Let $V$ be a nonsingular projective variety. Denote by $K_V$ a canonical divisor of $V$. One of the main strategies of explicit birational geometry is to find a practical constant $c = c(V)$ such that the pluricanonical map $\varphi_c$ or $\varphi_{-c}$ has the stable behavior (namely, being the Iitaka fibration).

♦ Today, I am concerned with 3-dimensional birational geometry. Assume $\dim(V) = 3$ and $\kappa := \kappa(V)$.

♦ Mori’s MMP $\Rightarrow$ $V$ has a minimal model $X$ (admitting $\mathbb{Q}$-factorial terminal singularities). Therefore the volume $\text{Vol}(V) = K_V^3$. 
Ueno, Kollár-Mori, Kawamata, Hacon-McKernan, Takayama, Tsuji, Viehweg-Zhang etc. $\Rightarrow$ existence of $c(V)$. 

Recently, J. A. Chen and M. Chen when $\kappa = 3$, $\text{Vol}(V) \geq \frac{1}{2660}$ and may take $c(V) = 73$. 

The aim of this talk is to introduce an application of our singularity basket consideration to $Q$-Fano 3-folds.
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♦ The aim of this talk is to introduce an application of our singularity basket consideration to \( \mathbb{Q} \)-Fano 3-folds.
♦ A 3-fold $X$ is called a terminal weak $\mathbb{Q}$-Fano 3-fold if $X$ has at worst terminal singularities and $-K_X$ is nef and big. $X$ is $\mathbb{Q}$-Fano if $-K_X$ is $\mathbb{Q}$-ample.

One motivation is Reid’s conjecture: $P_{-2} > 0$ for almost all $\mathbb{Q}$-Fano 3-folds. There are several known examples with $P_{-2} = 0$ by Fletcher and Altinok-Reid.

Another motivation is the optimal lower bound of $-K_X$ for $\mathbb{Q}$-Fano 3-folds. The boundedness has been proved by Kawamata ($\rho = 1$) and Kollár-Miyaoka-Mori-Takagi in general case.
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Definition and Motivations

♦ A 3-fold $X$ is called a **terminal weak $\mathbb{Q}$-Fano 3-fold** if $X$ has at worst terminal singularities and $-K_X$ is nef and big. $X$ is **$\mathbb{Q}$-Fano** if $-K_X$ is $\mathbb{Q}$-ample.

♦ **One motivation** is Reid’s conjecture: $P_{-2} > 0$ for almost all $\mathbb{Q}$-Fano 3-folds. There are several known examples with $P_{-2} = 0$ by Fletcher and Altinok-Reid.

♦ **Another motivation** is the optimal lower bound of $-K^3$ for $\mathbb{Q}$-Fano 3-folds. **The boundedness has been proved** by Kawamata ($\rho = 1$) and Kollár-Miyaoka-Mori-Takagi in general case.
Main Results

Theorem

Let $X$ be a terminal weak $\mathbb{Q}$-Fano 3-fold. Then

(i) $P_{-4} > 0$ with possibly one exception of a basket of singularities;
(ii) $P_{-6} > 0$ and $P_{-8} > 1$;
(iii) $-K_X^3 \geq \frac{1}{330}$. Furthermore $-K_X^3 = -\frac{1}{330}$ if and only if the virtual basket of singularities is

$$\left\{ \frac{1}{2}(1, -1, 1), \frac{1}{5}(1, -1, 2), \frac{1}{3}(1, -1, 1), \frac{1}{11}(1, -1, 2) \right\}.$$

Brown and Suzuki proved a sharp lower bound of $-K^3$ for certain special $\mathbb{Q}$-Fano 3-folds.
Reid’s Riemann-Roch Formula

For a $\mathbb{Q}$-factorial terminal 3-fold $X$, there is a unique singularity basket $\mathcal{B}(X)$ such that, for all $m > 1$,

$$\chi(\mathcal{O}_X(mK_X)) = \frac{1}{12} m(m - 1)(2m - 1)K_X^3 - (2m - 1)\chi(\mathcal{O}_X) + l(m)$$

where the correction term $l(m)$ can be computed as:

$$l(m) := \sum_{Q \in \mathcal{B}(X)} l_Q(m) := \sum_{Q \in \mathcal{B}(X)} \sum_{j=1}^{m-1} \frac{j b_Q (r_Q - \overline{jb_Q})}{2r_Q}$$

where the sum $\sum_Q$ runs through all singularity $Q \in \mathcal{B}(X)$ of type $\frac{1}{r_Q} (1, -1, b_Q)$ and $\overline{jb_Q}$ means the smallest residue of $jb_Q \mod r_Q$. 
A basket $B$ of singularities is a collection (permitting weights) of terminal quotient singularities of type $\frac{1}{r_i}(1, -1, b_i)$, $i \in I$ where $I$ is a finite index set. A single basket means a single singularity $Q$ of type $\frac{1}{r}(1, -1, b)$. For simplicity, we will always denote a single basket by $\{(b, r)\}$ or $(b, r)$. So we will write a basket as:

$$B := \{n_i \times (b_i, r_i)|i \in J, \ n_i \in \mathbb{Z}^+\}.$$
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$$B := \{n_i \times (b_i, r_i)| i \in J, \ n_i \in \mathbb{Z}^+\}.$$
Given a basket $B = \{(b_i, r_i) | i \in I\}$ and a fixed integer $n > 0$, we define

$$\Delta^n(B) := \sum_{i \in I} \Delta^n(b_i, r_i)$$

where

$$\Delta^n(b_i, r_i) := \frac{n b_i (r_i - n b_i)}{2 r_i} - \frac{n b_i (r_i - n b_i)}{2 r_i}.$$ 

Define $\sigma(B) := \sum_{i \in I} b_i$ and $\sigma'(B) := \sum_{i \in I} \frac{b_i^2}{r_i}.$
♦ Given a generalized basket

\[ B = \{(b_1, r_1), (b_2, r_2), \cdots, (b_k, r_k)\}, \]

we call the basket

\[ B' := \{(b_1 + b_2, r_1 + r_2), (b_3, r_3), \cdots, (b_k, r_k)\} \]

a packing of \( B \), written as \( B \succ B' \). (The symbol \( B \succ B' \) means either \( B \succ B' \) or \( B = B' \).)
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If, furthermore, \( b_1 r_2 - b_2 r_1 = 1 \), we call \( B \succ B' \) a prime packing.
Lemma

Assume $B \succeq B'$. Then

i. $\sigma(B) = \sigma(B')$ and $\sigma'(B) \geq \sigma'(B')$;

ii. For all integer $n > 1$, $\Delta^n(B) \geq \Delta^n(B')$;
Given a basket $B$. Set $S^{(0)} := \{ \frac{1}{n} | n \geq 2 \}$, $S^{(5)} := S^{(0)} \cup \{ \frac{2}{5} \}$ and inductively for all $n \geq 5$,

$$S^{(n)} := S^{(n-1)} \cup \left\{ \frac{b}{n} \mid 0 < b < \frac{n}{2}, \ b \text{ coprime to } n \right\}.$$
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$$S^{(n)} := S^{(n-1)} \cup \left\{\frac{b}{n} \mid 0 < b < \frac{n}{2}, \ b \text{ coprime to } n\right\}.$$  

The set $S^{(n)}$ then gives a division of the interval $(0, \frac{1}{2}] = \bigcup_i [\omega_i^{(n)}, \omega_{i+1}^{(n)}]$ with $\omega_i^{(n)}, \omega_{i+1}^{(n)} \in S^{(n)}$. Let $\omega_{i+1}^{(n)} = \frac{q_{i+1}}{p_{i+1}}$ and $\omega_i^{(n)} = \frac{q_i}{p_i}$ with $\text{g.c.d}(q_l, p_l) = 1$ for $l = i, i + 1$.

**Lemma**

*For all $n$ and $i$, $q_ip_{i+1} - p_iq_{i+1} = 1$.***
Given a basket $B = \{(b_i, r_i) | i = 1, \ldots, t\}$, for each $B_i = \{(b_i, r_i)\} \subset B$, if $\frac{b_i}{r_i} \in S^{(n)}$, we set $B_i^{(n)} := \{(b_i, r_i)\}$. If $\frac{b_i}{r_i} \not\in S^{(n)}$, then $\omega^{(n)}_{l+1} < \frac{b_i}{r_i} < \omega^{(n)}_l$ for some $l$. We write $\omega^{(n)}_l = \frac{q_l}{p_l}$ and $\omega^{(n)}_{l+1} = \frac{q_{l+1}}{p_{l+1}}$ respectively. In this situation, we can unpack the $B_i$ to $B_i^{(n)} := \{(r_i q_l - b_i p_l) \times (q_{l+1}, p_{l+1}), (-r_i q_{l+1} + b_i p_{l+1}) \times (q_l, p_l)\}$. Taking the union of those $B_i^{(n)}$ with corresponding multiplicities, we get a new basket $B^{(n)}(B)$. 
Canonical sequence of $B$

Lemma

\[ B^{(n-1)}(B) = B^{(n-1)}(B^{(n)}(B)) \succ B^{(n)}(B) \]

for all $n \geq 1$.

Thus we have the chain of baskets:

\[ B^{(0)}(B) \succ B^{(5)}(B) \succ \ldots \succ B^{(n)}(B) \succ \ldots \succ B. \]

(The Canonical Sequence)
Each step $B^{(n-1)}(B) \succ B^{(n)}(B)$ can be achieved by certain number ($\epsilon_n(B)$) of prime packings of type
\[
\{(b_1, r_1), (b_2, r_2)\} \succ \{(b_1 + b_2, r_1 + r_2)\}
\]
with $r_1 + r_2 = n$. The number $\epsilon_n(B)$ can be computed with the following:

**Lemma**

For the sequence $\{B^{(n)}(B)\}$, one has:

(i) $\Delta^j(B^{(0)}(B)) = \Delta^j(B)$ for $j = 3, 4$;

(ii) $\Delta^j(B^{(n-1)}(B)) = \Delta^j(B^{(n)}(B))$ for all $j < n$;

(iii) $\Delta^n(B^{(n-1)}(B)) = \Delta^n(B^{(n)}(B)) + \epsilon_n(B)$. 
Reid’s Formula \( \implies \chi(\mathcal{O}_X(mK_X)) \) and \( K^3 \) are determined by \( \mathcal{B}(X), \chi(\mathcal{O}_X) \) and \( \chi_2 := \chi(\mathcal{O}_X(2K_X)). \)
Reid’s Formula $\Rightarrow \chi(\mathcal{O}_X(mK_X))$ and $K^3$ are determined by $\mathcal{B}(X)$, $\chi(\mathcal{O}_X)$ and $\chi_2 := \chi(\mathcal{O}_X(2K_X))$.

Assume that $B$ is a basket, $\tilde{\chi}$ and $\tilde{\chi}_2$ are integers. We call the triple $\mathcal{B} := (B, \tilde{\chi}, \tilde{\chi}_2)$ a formal basket.
First we define

\[
\begin{align*}
\chi_2(B) & := \tilde{\chi}_2, \\
\chi_3(B) & := -\sigma(B) + 10\tilde{\chi} + 5\tilde{\chi}_2
\end{align*}
\]

and the volume

\[
K^3(B) := -\sigma + \sigma' + 6\tilde{\chi} + 2\tilde{\chi}_2.
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\]

For \( m \geq 4 \), the Euler characteristic \( \chi_m(B) \) is defined inductively by

\[
\chi_{m+1}(B) - \chi_m(B) := \frac{m^2}{2}(K^3(B) - \sigma'(B)) + \frac{m}{2}\sigma(B) - 2\tilde{\chi} + \Delta^m(B).
\]
Given a \( \mathbb{Q} \)-factorial canonical 3-fold \( X \), one can associate to \( X \) a formal basket \( \mathcal{B}(X) := (B, \tilde{\chi}, \tilde{\chi}_2) \) where \( B = \mathcal{B}(X) \), \( \tilde{\chi} = \chi(\mathcal{O}_X) \) and \( \tilde{\chi}_2 = \chi_2(X) \). Then one can see:

\[
K_X^3 = K^3(\mathcal{B});
\]

\[
\chi(\mathcal{O}_X(mK_X)) = \chi_m(\mathcal{B}).
\]
Given a $\mathbb{Q}$-factorial canonical 3-fold $X$, one can associate to $X$ a formal basket $\mathcal{B}(X) := (B, \tilde{\chi}, \tilde{\chi}_2)$ where $B = \mathcal{B}(X)$, $\tilde{\chi} = \chi(O_X)$ and $\tilde{\chi}_2 = \chi_2(X)$. Then one can see:

$$K_X^3 = K^3(\mathcal{B});$$

$$\chi(O_X(mK_X)) = \chi_m(\mathcal{B}).$$

Let $\mathcal{B} := (B, \tilde{\chi}, \tilde{\chi}_2)$ and $\mathcal{B}^\prime := (B^\prime, \tilde{\chi}, \tilde{\chi}_2)$ be two formal baskets. We say that $\mathcal{B}^\prime$ is a packing of $\mathcal{B}$ (written as $\mathcal{B} \succ \mathcal{B}^\prime$) if $B \succ B^\prime$. Clearly “packing” between formal baskets gives a partial ordering.
Lemma

Assume $\mathbf{B} := (B, \tilde{\chi}, \tilde{\chi}_2) \succ \mathbf{B}' := (B', \tilde{\chi}, \tilde{\chi}_2)$. Then:

1. $K^3(\mathbf{B}) \geq K^3(\mathbf{B}')$;
2. $\chi_m(\mathbf{B}) \geq \chi_m(\mathbf{B}')$ for all $m \geq 2$.

♦ The Main Strategy is to find two computable formal baskets $\mathbf{B}_1$ and $\mathbf{B}_2$ such that $\mathbf{B}_1 \succ \mathbf{B}(X) \succ \mathbf{B}_2$. 

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Let $X$ be a weak $\mathbb{Q}$-Fano 3-fold. Then $\chi(O_X) = 1$ (constant). For $m > 0$, by duality, $P_{-m} = \chi(O_X(-mK_X)) = -\chi(O_X((m+1)K_X))$. We will study the formal basket $B(X) := \{B(X), 1, P_{-1}\}$. 
Let $X$ be a weak $\mathbb{Q}$-Fano 3-fold. Then $\chi(\mathcal{O}_X) = 1$ (constant). For $m > 0$, by duality, $P_{-m} = \chi(\mathcal{O}_X(-mK_X)) = -\chi(\mathcal{O}_X((m + 1)K_X))$.

We will study the formal basket $B(X) := \{ B(X), 1, P_{-1} \}$.

More generally, we will consider a formal basket $B = (B, 1, P_{-1})$. We begin by computing the non-negative number $\epsilon_n$ and $B^{(0)}, B^{(5)}$ in terms of $P_{-m}$.
From the definition of $P_{-m}$ we get:

$$\sigma(B) = 10 - 5P_{-1} + P_{-2},$$

$$\Delta^{m+1} = (2 - 5(m+1) + 2(m+1)^2) + \frac{1}{2}(m+1)(2 - 3m)P_{-1} + \frac{1}{2}m(m+1)P_{-2} + P_{-m} - P_{-(m+1)}.$$
From the definition of $P_{-m}$ we get:

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$$+ \frac{1}{2}m(m + 1)P_{-2} + P_{-m} - P_{-(m+1)}.$$

In particular, we have:

$$\Delta^3 = 5 - 6P_{-1} + 4P_{-2} - P_{-3};$$

$$\Delta^4 = 14 - 14P_{-1} + 6P_{-2} + P_{-3} - P_{-4};$$
Assume $B^{(0)}(B) = \{ n_{1,r}^0 \times (1, r) | r \geq 2 \}$. By calculations, we have:

$$\sigma(B) = \sigma(B^{(0)}(B)) = \sum n_{1,r}^0;$$

$$\Delta^3(B) = \Delta^3(B^{(0)}(B)) = n_{1,2}^0;$$

$$\Delta^4(B) = \Delta^4(B^{(0)}(B)) = 2n_{1,2}^0 + n_{1,3}^0.$$
Assume $B^{(0)}(B) = \{ n_{1,r}^0 \times (1, r) | r \geq 2 \}$. By calculations, we have:

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$$\Delta^4(B) = \Delta^4(B^{(0)}(B)) = 2n_{1,2}^0 + n_{1,3}^0.$$

Thus one gets $B^{(0)}$ as follows:

\[
\begin{align*}
    n_{1,2}^0 &= 5 - 6P_{-1} + 4P_{-2} - P_{-3} \\
    n_{1,3}^0 &= 4 - 2P_{-1} - 2P_{-2} + 3P_{-3} - P_{-4} \\
    n_{1,4}^0 &= 1 + 3P_{-1} - P_{-2} - 2P_{-3} + P_{-4} - \sigma_5 \\
    n_{1,r}^0 &= n_{1,r}^0, \quad r \geq 5,
\end{align*}
\]
A computation gives:

\[ \varepsilon_5 = 2 + P_{-2} - 2P_{-4} + P_{-5} - \sigma_5. \]
A computation gives:

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Therefore we get \( B^{(5)} \) as follows:

\[
\begin{align*}
\forall 1,2 & : n_{1,2}^5 = 3 - 6P_{-1} + 3P_{-2} - P_{-3} + 2P_{-4} - P_{-5} + \sigma_5 \\
\forall 2,5 & : n_{2,5}^5 = 2 + P_{-2} - 2P_{-4} + P_{-5} - \sigma_5 \\
\forall 1,3 & : n_{1,3}^5 = 2 - 2P_{-1} - 3P_{-2} + 3P_{-3} + P_{-4} - P_{-5} + \sigma_5 \\
\forall 1,4 & : n_{1,4}^5 = 1 + 3P_{-1} - P_{-2} - 2P_{-3} + P_{-4} - \sigma_5 \\
\forall 1,r & : n_{1,r}^5 = n_{1,r}^0, r \geq 5
\end{align*}
\]
Since $B^{(5)} = B^{(6)}$, we see $\epsilon_6 = 0$ and on the other hand

$$\epsilon_6 = 3P_{-1} + P_{-2} - P_{-3} - P_{-4} - P_{-5} + P_{-6} - \epsilon = 0,$$

where $\epsilon := 2\sigma_5 - n_{1,5}^0 \geq 0$. 

Applications to Weak $\mathbb{Q}$-Fano 3-folds
Since $B^{(5)} = B^{(6)}$, we see $\epsilon_6 = 0$ and on the other hand

$$\epsilon_6 = 3P_{-1} + P_{-2} - P_{-3} - P_{-4} - P_{-5} + P_{-6} - \epsilon = 0,$$

where $\epsilon := 2\sigma_5 - n_{1,5}^0 \geq 0$.

By a similar calculation, one gets:

$$\epsilon_7 = 1 + P_{-1} + P_{-2} - P_{-5} - P_{-6} + P_{-7} - 2\sigma_5 + 2n_{1,5}^0 + n_{1,6}^0$$

$$\epsilon_8 = 2P_{-1} + P_{-2} + P_{-3} - P_{-4} - P_{-5} - P_{-7} + P_{-8} - 3\sigma_5 + 3n_{1,5}^0 + 2n_{1,6}^0 + n_{1,7}^0$$
Miyaoka-Reid inequality:

\[ \gamma(B) := \sum_{i=1}^{t} \frac{1}{r_i} - \sum_{i=1}^{t} r_i + 24 \geq 0. \]
Miyaoka-Reid inequality:

\[ \gamma(B) := \sum_{i=1}^{t} \frac{1}{r_i} - \sum_{i=1}^{t} r_i + 24 \geq 0. \]

\(-K^3(B) = -K_X^3 > 0\) gives the inequality:

\[ \sigma'(B) < 2P_{-1} + \sigma(B) - 6. \]
Miyaoka-Reid inequality:

$$\gamma(B) := \sum_{i=1}^{t} \frac{1}{r_i} - \sum_{i=1}^{t} r_i + 24 \geq 0.$$ 

$-K^3(B) = -K^3_X > 0$ gives the inequality:

$$\sigma'(B) < 2P_{-1} + \sigma(B) - 6.$$ 

Whenever $P_{-m} > 0$ and $P_{-n} > 0$, one has

$$P_{-m-n} \geq P_{-m} + P_{-n} - 1.$$
Proposition

Given \( p_i \in \mathbb{Z}^+ \), there are only finitely many formal baskets admitting \((P_{-1}, P_{-2}) = (p_1, p_2)\) and satisfying Miyaoka-Reid inequality.

Corollary

There are finitely many formal baskets satisfying \( P_{-2}(B) = 0 \).
**Theorem**

Any geometric basket with $P_{-2} = 0$ is among the following list:

<table>
<thead>
<tr>
<th>$B$</th>
<th>$-K^3$</th>
<th>$P_{-3}$</th>
<th>$P_{-4}$</th>
<th>$P_{-5}$</th>
<th>$P_{-6}$</th>
<th>$P_{-7}$</th>
<th>$P_{-8}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No.1. ${2 \times (1, 2), 3 \times (2, 5), (1, 3), (1, 4)}$</td>
<td>1/60</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>No.2. ${5 \times (1, 2), 2 \times (1, 3), (2, 7), (1, 4)}$</td>
<td>1/84</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>No.3. ${5 \times (1, 2), 2 \times (1, 3), (3, 11)}$</td>
<td>1/66</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>No.4. ${5 \times (1, 2), (1, 3), (3, 10), (1, 4)}$</td>
<td>1/60</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>No.5. ${5 \times (1, 2), (1, 3), 2 \times (2, 7)}$</td>
<td>1/42</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>No.6. ${4 \times (1, 2), (2, 5), 2 \times (1, 3), 2 \times (1, 4)}$</td>
<td>1/30</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>No.7. ${3 \times (1, 2), (2, 5), 5 \times (1, 3)}$</td>
<td>1/30</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>No.8. ${2 \times (1, 2), (3, 7), 5 \times (1, 3)}$</td>
<td>1/21</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>No.9. ${(1, 2), (4, 9), 5 \times (1, 3)}$</td>
<td>1/18</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>No.10. ${3 \times (1, 2), (3, 8), 4 \times (1, 3)}$</td>
<td>1/24</td>
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<td>No.11. ${3 \times (1, 2), (4, 11), 3 \times (1, 3)}$</td>
<td>1/22</td>
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<tr>
<td>No.12. ${3 \times (1, 2), (5, 14), 2 \times (1, 3)}$</td>
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<td>No.13. ${2 \times (1, 2), 2 \times (2, 5), 4 \times (1, 3)}$</td>
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<td>No.14. ${(1, 2), (3, 7), (2, 5), 4 \times (1, 3)}$</td>
<td>17/210</td>
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<td>No.15. ${2 \times (1, 2), (2, 5), (3, 8), 3 \times (1, 3)}$</td>
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<td>No.16. ${2 \times (1, 2), (5, 13), 3 \times (1, 3)}$</td>
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<td>No.17. ${(1, 2), 3 \times (2, 5), 3 \times (1, 3)}$</td>
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<td>No.18. ${4 \times (1, 2), 5 \times (1, 3), (1, 4)}$</td>
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<tr>
<td>No.19. ${4 \times (1, 2), 4 \times (1, 3), (2, 7)}$</td>
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<td>No.20. ${4 \times (1, 2), 3 \times (1, 3), (3, 10)}$</td>
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<td>No.21. ${3 \times (1, 2), (2, 5), 4 \times (1, 3), (1, 4)}$</td>
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<td>No.22. ${3 \times (1, 2), 7 \times (1, 3)}$</td>
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<tr>
<td>No.23. ${2 \times (1, 2), (2, 5), 6 \times (1, 3)}$</td>
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</tbody>
</table>
Theorem

Let $X$ be a terminal weak $\mathbb{Q}$-Fano 3-fold. Then $P_{-6} > 0$.

Proof.

Set $B := B_X$. If $P_{-6} = 0$, then $P_{-1} = P_{-2} = P_{-3} = 0$. By $\epsilon_6 = 0$, we get $P_{-4} = P_{-5} = \epsilon = 0$. Thus $B^{(5)}(B) = \{3 \times (1, 2), 2 \times (2, 5), 2 \times (1, 3), (1, 4)\}$. The above formulae will give a contradiction. Thus we know that such a basket $B$ does not exist. Thus $P_{-6}(X) > 0$. 
Theorem

Let $X$ be a terminal weak $\mathbb{Q}$-Fano 3-fold. Then $P_{-4} > 0$ unless $B_X = \{2 \times (1, 2), 3 \times (2, 5), (1, 3), (1, 4)\}$.

Theorem

Let $X$ be a terminal weak $\mathbb{Q}$-Fano 3-fold. Then $P_{-8} \geq 2$. 
As a direct result of the Riemann-Roch formula, we have:

$$\frac{1}{2}(-K^3) = P_{-1} - 3 + l(2),$$

$$\frac{5}{2}(-K^3) = P_{-2} - 5 + l(3).$$
As a direct result of the Riemann-Roch formula, we have:

\[ \frac{1}{2}(-K^3) = P_{-1} - 3 + l(2), \]
\[ \frac{5}{2}(-K^3) = P_{-2} - 5 + l(3). \]

When \( P_{-1} \) or \( P_{-2} \) is bigger, we have effective lower bound of \( -K_X^3 \).
As a direct result of the Riemann-Roch formula, we have:

\[
\frac{1}{2}(-K^3) = P_{-1} - 3 + l(2),
\]

\[
\frac{5}{2}(-K^3) = P_{-2} - 5 + l(3).
\]

When \( P_{-1} \) or \( P_{-2} \) is bigger, we have effective lower bound of \(-K_X^3\).

Otherwise, we have seen \( B \) has finitely many possibilities. We can work out a detailed classification.
Assume $P_{-1}$ and $P_{-2}$ are small. We can calculate $B^{(0)}$. Since $B^{(0)} \gtrsim B^{(5)} \gtrsim B^{(3)}$, we have $K^{3}(B^{(0)}) \geq K^{3}(B^{(5)}) \geq K^{3}(X)$ and thus $-K^{3}(X) \geq -K^{3}(B^{(5)}) \geq -K^{3}(B^{(0)})$. Detailed classification tells us what $B^{(0)}$ or $B^{(5)}$ is. Thus we are able to get an effective lower bound of $-K^{3}(X)$.  

Meng Chen Fudan University, Shanghai  

An Optimal Boundedness for Weak $\mathbb{Q}$-Fano Threefolds
Assume $P_{-1}$ and $P_{-2}$ are small. We can calculate $B^{(0)}$.

Since $B^{(0)} \gtrless B^{(5)} \gtrless B$, we have

$$K^3(B^{(0)}) \geq K^3(B^{(5)}) \geq K^3_X$$

and thus

$$-K^3_X \geq -K^3(B^{(5)}) \geq -K^3(B^{(0)})$$
Assume $P_{-1}$ and $P_{-2}$ are small. We can calculate $B^{(0)}$.

Since $B^{(0)} \succeq B^{(5)} \succeq B$, we have $K^3(B^{(0)}) \geq K^3(B^{(5)}) \geq K^3_X$ and thus

$$-K^3_X \geq -K^3(B^{(5)}) \geq -K^3(B^{(0)}).$$

Detailed classification tells us what $B^{(0)}$ or $B^{(5)}$ is. Thus we are able to get an effective lower bound of $-K^3_X$. 

Meng Chen  
Fudan University, Shanghai  
An Optimal Boundedness for Weak $\mathbb{Q}$-Fano Threefolds
Theorem

Let $X$ be a terminal weak $\mathbb{Q}$-Fano 3-fold. Then:

$$-K^3(X) \geq \frac{1}{330}$$

and equality holds if, and only if, $B = \{(1, 2), (2, 5), (1, 3), (2, 11)\}$. 
Example

(Fletcher-Reid) The general hyper-surface

\[ X_{66} \subset \mathbb{P}(1, 5, 6, 22, 33) \]

has anti-volume \(-K_X^3 = \frac{1}{330}\) and

\[ B_X = \{(1, 2), (2, 5), (1, 3), (2, 11)\} \].

♦ Thanks very much.