Quantum Invariance of Flops

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MOTIVATION.

In 1988, Mori established the three dimensional Minimal Model Program, which successfully provides a way to get minimal models in a fixed birational equivalence class.

However minimal models in a fixed birational equivalence class are not unique, so the most important thing is to find invariants among minimal models.

For this purpose, Chin-Lung Wang created the notion of $K$-equivalent varieties to generalize the one of minimal models.
Definition

Two ($\mathbb{Q}$-Gorenstein) varieties $X$ and $X'$ are $K$-equivalent if there exist birational morphisms $\phi : Y \to X$ and $\phi' : Y \to X'$ with $Y$ smooth such that

$$\phi^* K_X = \phi'^* K_{X'}.$$

(eg. birational Calabi-Yau's manifolds or minimal models.)
Batyrev and C.-L. Wang showed that $K$-equivalent smooth varieties have the same Betti numbers.

However, the cohomology ring structures are in general different.

Two natural questions arise here:

1. Is there a *canonical correspondence* between the cohomology groups of $K$-equivalent smooth varieties?
2. Is there a *modified ring structure* which is invariant under the $K$-equivalence relation?
CONJECTURE

There exists $\mathcal{F} = [\tilde{f}] + \sum T_i \in A^d(X \times X')$ which gives isomorphism of Chow motives $[X] \cong [X']$. $\mathcal{F}$ is orthogonal (preserving the Poincaré pairing) and

$$\mathcal{F} : QH(X) \cong QH(X')$$

after an analytic continuation over the Kähler moduli. (In general $X$ and $X'$ are not even homotopy equivalent.)
**Gromov-Witten invariants:** For $\alpha \in H(X)^{\otimes n}$, $\beta \in H_2(X, \mathbb{Z})$

$$\langle \alpha \rangle_{g,n,\beta} = \int_{[M_{g,n}(X,\beta)]^{\text{vir}}} \text{ev}^* \alpha$$

with $\text{ev} = \prod e_i : M_{g,n}(X,\beta) \to X^n$ being the evaluation map.

**Big quantum ring:** Let $\{ T_i \}$ be a basis of $H(X)$ and $t = \sum t_i T_i$,

$$F_g(t) := \sum_{n,\beta} \frac{q^\beta}{n!} \langle t^n \rangle_{g,n,\beta}.$$ 

The quantum product uses only $g = 0$. Let $\Phi = F_0$,

$$T_i *_t T_j = \sum_k \Phi_{ijk}(t) T^k = \sum_{k,n,\beta} \frac{q^\beta}{n!} \langle T_i, T_j, T_k, t^n \rangle_{0,n+3,\beta} T^k,$$

where $g_{ij} = (T_i, T_j)$, $T^j = g^{ij} T_i$ is the dual basis.
**Kähler moduli:** Let $\mathcal{K}_X^\mathbb{C} = H^{1,1}_\mathbb{R}(X) \times \mathcal{K}_X$ be the complexified Kähler cone and let $\omega = B + iH \in \mathcal{K}_X^\mathbb{C}$. Then

$$q_\beta = e^{2\pi i(\omega,\beta)}, \quad |q_\beta| = e^{-2\pi(H.\beta)} < 1.$$  

It is conjectured that $\langle \alpha \rangle = \sum \langle \alpha \rangle_\beta q_\beta$ converges in $\omega \in \mathcal{K}_X^\mathbb{C}$.

**Analytic continuation:** For $X =_K X'$ and $X \not\cong X'$, $H^2(X) \cong H^2(X')$ but $\mathcal{K}_X \cap \mathcal{K}_{X'} = \emptyset$ in $H^2$. If $\mathcal{F}$ preserves the Poincaré pairing, then $\mathcal{F}(T_i * t T_j) = \mathcal{F} T_i * \mathcal{F} t \mathcal{F} T_j$ is equivalent to

$$\Phi_{ijk}^X(\omega, t) = \Phi_{ijk}^{X'}(\mathcal{F} \omega, \mathcal{F} t).$$

up to analytic continuations in $\omega$ from $\mathcal{K}_X^\mathbb{C}$ to $\mathcal{K}_{X'}^\mathbb{C}$. Since $\omega$ and $\mathcal{F} \omega$ are canonically identified and $(\omega, \beta)_X = (\mathcal{F} \omega, \mathcal{F} \beta)_{X'}$, formally this means

$$q_\beta \mapsto q^{\mathcal{F} \beta}.$$
Ordinary $\mathbb{P}^r$-Flop Diagram

$X$ : smooth projective, $\psi$ : log-extremal small contraction, $Z$, $S$: $\psi$-exceptional sets, $F$ : a vector bundle of rank $r + 1$, $N_{Z/X}|_{Z_s} \cong \mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus (r+1)}$,

$E = \mathbb{P}_S(F) \times_S \mathbb{P}_S(F') \xrightarrow{j} Y$

$Z = \mathbb{P}_S(F) \xrightarrow{i} X$

$Z' = \mathbb{P}_S(F') \xrightarrow{i'} X'$

$F'$ : a vector bundle of rank $r + 1$, $N_{Z'/X'}|_{Z'_s} \cong \mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus (r+1)}$.

If $S$ is a point, then we call it a simple ordinary $\mathbb{P}^r$-flop.

(1) For $\mathbb{P}^r$ flops $f : X \to X'$, the graph closure $\mathcal{F} = [\Gamma_f]$ induces canonical isomorphism of Chow motives.

(2) For simple $\mathbb{P}^r$ flops, the full Gromov-Witten theory in the stable range $2g + n \geq 3$ can be analytic continued to each other under the graph correspondence.

(3) For $\mathbb{P}^r$ flops, the Gromov-Witten theory in the stable range $2g + n \geq 3$ attached the extremal rays are invariant up to analytic continuations.

(4) For $\mathbb{P}^r$ flops with split bundles $F = \bigoplus L_i$ and $F' = \bigoplus L'_i$, the big quantum cohomology rings are analytic continuations of each other under the graph correspondence.
**Genus zero theory:** The Conjecture for 3-folds was previously solved by A. Li and Y. Ruan in 1998. 3 ingredients of their proof:

1. Symplectic deformations and decompositions of $K$ equivalent maps into $\mathbb{P}^1$ flops. (Kawamata, Kollár, Friedman.)

2. Multiple cover formula for $\mathbb{P}^1 = C \subset X$, $N_{C/X} = \mathcal{O}(-1)^{\oplus 2}$:

   $$\langle - \rangle_{0, dC}^X = \frac{1}{d^3}.$$

   (Aspinwall-Morrison, Voisin, Lian-Liu-Yau.)

3. Relative GW invariants and the degeneration formula. (Li-Ruan, Inoel-Parker, J. Li.) For $\beta \not\in \mathbb{Z}[C]$,

   $$\langle \alpha_1, \ldots, \alpha_n \rangle_{g, n, \beta} = \langle \mathcal{F}\alpha_1, \ldots, \mathcal{F}\alpha_n \rangle_{g, n, \mathcal{F}\beta}.$$
We make progresses on (2) and (3).

Quantum Product

Recall that the \textit{big quantum product} is defined by

\[ T_i \ast T_j = \sum_k \Phi_{ijk} T^k \]

where

\[
\Phi_{ijk} = \sum_{n=0}^{\infty} \sum_{\beta \in A_1(X)} \frac{1}{n!} \langle T_i, T_j, T_k, \gamma^n \rangle_\beta q^\beta.
\]

The \( n = 0 \) part \( \Phi_{ijk}(0) \) gives the \textit{small quantum product}, that is,

\[
T_i \ast T_j = \sum_k \sum_{\beta \in A_1(X)} \langle T_i, T_j, T_k \rangle_\beta q^\beta T^k.
\]
Three Point Functions

The genus zero three point functions (as formal power series)

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle : = \sum_{\beta \in A_1(X)} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,3,\beta} q^\beta$$

$$= (\alpha_1.\alpha_2.\alpha_3)$$

$$+ \sum_{d \in \mathbb{N}} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_d q^{d\ell} + \sum_{\beta \notin \mathbb{Z}\ell} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_\beta q^\beta.$$

together with the Poincaré pairing $(-,-)$ determine the small quantum product.
The Strategy

For $\beta = 0$, we derive the defect of triple products precisely.

For $\beta = d \ell$, $\langle \alpha_1, \ldots, \alpha_n \rangle_{0,n,d \ell} = (-1)^{d-1}(r+1) N_{l_1,\ldots,l_n} d^{n-3}$.

To determine $\langle \alpha_1, \ldots, \alpha_n \rangle = \sum_d \langle \alpha_1, \ldots, \alpha_n \rangle_{0,n,d \ell} (q^\ell)^d$ and then compare $\langle \alpha_1, \ldots, \alpha_n \rangle^X$ with $\langle \mathcal{F} \alpha_1, \ldots, \mathcal{F} \alpha_n \rangle^{X'}$ directly.

For $\beta \neq d \ell$,

Absolute invariants to

Relative invariants

Relative invariants to

Absolute invariants

H.-W. Lin, Y.-P. Lee and C.-L. Wang
Quantum Invariance of Flops
The Defect of Product Structure

\( f : X \rightarrow X' \) a simple \( \mathbb{P}^r \) flop, \( h = \) hyperplane class of \( Z = \mathbb{P}^r \).

**Theorem** For simple \( \mathbb{P}^r \)-flops, \( \alpha \in A^1(X), \beta \in A^2(X), \gamma \in A^3(X) \) with \( l_1 \leq l_2 \leq l_3 \leq r, \ l_1 + l_2 + l_3 = \dim X = 2r + 1 \),

\[
\mathcal{F}\alpha.\mathcal{F}\beta.\mathcal{F}\gamma = \alpha.\beta.\gamma + (-1)^r (\alpha.h^{r-l_1})(\beta.h^{r-l_2})(\gamma.h^{r-l_3}).
\]
Generalized multiple cover formula

**Theorem**
\[ \forall \alpha_i \in A^l_i(X) \text{ with } 1 \leq l_i \leq r \text{ and } \sum_{i=1}^{n} l_i = 2r + 1 + (n - 3), \]
there are recursively determined universal constants \( N_{l_1,...,l_n} \), which is independent of \( d \), such that for \( n \leq 3 \), \( N_* \equiv 1 \) and

\[
\langle \alpha_1, \ldots, \alpha_n \rangle_{0,n,d} \equiv \int_{[\tilde{M}_{0,n}(X,d\ell)]^{virt}} e_1^*\alpha_1 \cdots e_n^*\alpha_n \\
= (-1)^{(d-1)(r+1)} N_{l_1,...,l_n} d^{n-3} (\alpha_1.h^{r-l_1}) \cdots (\alpha_n.h^{r-l_n}).
\]
Quantum Corrections by Extremal Rays

Theorem
1. *The three point functions attached to the extremal ray exactly remedy the defect caused by the classical product.*
2. *The big quantum product on exceptional classes is also invariant under simple ordinary flops.*

\[
\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{\text{extremal}} = \sum_{d \in \mathbb{N}} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{d \ell} q^{d\ell}
\]

\[
\langle \mathcal{F}_\alpha_1, \mathcal{F}_\alpha_2, \mathcal{F}_\alpha_3 \rangle_{\text{extremal}} = \sum_{d \in \mathbb{N}} \langle \mathcal{F}_\alpha_1, \mathcal{F}_\alpha_2, \mathcal{F}_\alpha_3 \rangle_{d \ell'} q^{d\ell'}
\]

and

\[
\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{\text{extremal}} - \langle \mathcal{F}_\alpha_1, \mathcal{F}_\alpha_2, \mathcal{F}_\alpha_3 \rangle_{\text{extremal}} = (-1)^r (\alpha.h^{r-h_1})(\beta.h^{r-h_2})(\gamma.h^{r-h_3}) = (\mathcal{F}_\alpha_1.\mathcal{F}_\alpha_2.\mathcal{F}_\alpha_3) - (\alpha_1.\alpha_2.\alpha_3).
\]
Generalized multiple cover formula with 3 points

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{0,n,d} \equiv \int_{[\bar{M}_{0,n}(X,d\ell)]_{\text{virt}}} e_1^* \alpha_1 \cdots e_n^* \alpha_n$$

$$= (-1)^{(d-1)(r+1)} N_{l_1,\ldots,l_n} d^{n-3}(\alpha_1.h^{r-l_1}) \cdots (\alpha_n.h^{r-l_n}).$$

When $n = 3$,

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,3,d} = (-1)^{(d-1)(r+1)} (\alpha_1.h^{r-l_1})(\alpha_2.h^{r-l_2})(\alpha_3.h^{r-l_3}).$$

Thus

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{I+II} := (\alpha_1.\alpha_2.\alpha_3) + \sum_{d \in \mathbb{N}} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{d\ell} q^{d\ell}$$

$$= (\alpha_1.\alpha_2.\alpha_3) + (\alpha_1.h^{r-l_1})(\alpha_2.h^{r-l_2})(\alpha_3.h^{r-l_3}) \frac{q^\ell}{1 + (-1)^r q^\ell}.$$
\( q^{\ell'} \sim q^{-\ell} \) in terms of Analytic Continuation

Let \( \omega \in \mathcal{K}_X \), the Kähler cone of \( X \) and \( q^\beta = e^{-2\pi(\omega, \beta)} \). Then \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{I+II}^X \) converges in \( \mathcal{K}_X \) and equals

\[
(\alpha_1 \cdot \alpha_2 \cdot \alpha_3) + (\alpha_1 \cdot h^{-l_1})(\alpha_2 \cdot h^{-l_2})(\alpha_3 \cdot h^{-l_3}) \frac{e^{-2\pi(\omega, \ell)}}{1 + (-1)^r e^{-2\pi(\omega, \ell)}}.
\]

Similarly, \( \langle \mathcal{F}\alpha_1, \mathcal{F}\alpha_2, \mathcal{F}\alpha_3 \rangle_{I+II}^{X'} \) converges in \( \mathcal{K}_X' \) and equals (here, we use \( (\mathcal{F}\alpha_i \cdot h'(r-l_i)) = (-1)^{l_i}(\mathcal{F}\alpha_i \cdot \mathcal{F}h^{-l_i}) = (-1)^{l_i}(\alpha_i \cdot h^{-l_i}). \))

\[
(\mathcal{F}\alpha_1 \cdot \mathcal{F}\alpha_2 \cdot \mathcal{F}\alpha_3) - (\alpha_1 \cdot h^{-l_1})(\alpha_2 \cdot h^{-l_2})(\alpha_3 \cdot h^{-l_3}) \frac{e^{-2\pi(\omega, \ell')}}{1 + (-1)^r e^{-2\pi(\omega, \ell')}},
\]

which is the analytic continuation of the previous one from \( \mathcal{K}_X \) to \( \mathcal{K}_X' \) via \( \ell' \sim -\ell \). Equivalently, \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{I+II} \) is well-defined on \( \mathcal{K}_X \cup \mathcal{K}_X' \), which verifies the functional equation

\[
\mathcal{F} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{I+II}^X \cong \langle \mathcal{F}\alpha_1, \mathcal{F}\alpha_2, \mathcal{F}\alpha_3 \rangle_{I+II}^{X'}.
\]
GW Invariants with Gravitational Descendent

In general, $L_i$ is the line bundle on $\bar{M}_{0,n}(\mathbb{P}^r, d)$ whose fiber at $(f; C, (x_1, \ldots, x_n))$ is the cotangent line $T^* x_i C$ and $\psi_i = c_1^G(L_i)$.

Multiple point functions with descendent are defined by

$$\left\langle \tau_{k_1}(h_1), \cdots, \tau_{k_n}(h_n) \right\rangle_d := \int_{\bar{M}_{0,n}(\mathbb{P}^r, d)} \left( \prod_{i=1}^n \psi_i^{k_i} e_i^* h_i \right) . e_T(U_d).$$
One Point Function with Descendent

**Theorem** For \( l + k = 2r - 1, \ 1 \leq l \leq r \),

\[
\langle \tau_k h^l \rangle_d = \frac{(-1)^{d(r+1)+k}}{d^{k+2}} C_{r}^{k+1}.
\]

The invariant is zero if \( l + k \neq 2r - 1 \) by the dimension count.

**Proof.**

We compare

\[
A := \int_{\mathbb{P}^r} h^l e_1^* \frac{e_T(U_d)}{z(z - \psi)} = \sum_{k \geq 0} \frac{1}{z^{k+2}} \langle \tau_k h^l \rangle_d
\]

with

\[
A = \int_{\mathbb{P}^r} h^l \frac{l_d^* Q_d}{\prod_{m=1}^{d} (h + mz)^{r+1}} = (-1)^{(d-1)(r+1)} \int_{\mathbb{P}^r} \frac{h^l}{(h + dz)^{r+1}}.
\]
For $L \in \text{Pic}(X)$ and $i \neq j$,

$$e_i^* L \cap [\bar{M}_{0,n}(X, \beta)]^{virt} = (e_j^* L + (\beta, L)\psi_j) \cap [\bar{M}_{0,n}(X, \beta)]^{virt} - \sum_{\beta_1 + \beta_2 = \beta} (\beta_1, L)[D_{i, \beta_1|j, \beta_2}]^{virt},$$

$$\psi_i + \psi_j = [D_{i|j}]^{virt},$$

where

$$D_{i|j} = \sum_{\beta_1 + \beta_2 = \beta} D_{i, \beta_1|j, \beta_2} = \sum_{\beta_1 + \beta_2 = \beta} \sum_{i \in A, j \in B; A \sqcup B = \{1, \ldots, n\}} D(A, B; \beta_1, \beta_2).$$
Two Point Function

**Theorem** The only non-trivial two-point function in degree $d \ell$ is given by

$$\langle h^r, h^r \rangle_d = (-1)^{(d-1)(r+1)} \frac{1}{d}.$$ 

Proof. We apply the divisor relation repeatedly to compute

$$\langle h^r, h^r \rangle_d = d \langle h^{r-1}, \tau_1(h^r) \rangle_d = \cdots = (r-1)^{d-1} \langle h, \tau_{r-1}(h^r) \rangle_d = d^r \langle \tau_{r-1}(h^r) \rangle_d.$$ 

where the last equality is by the divisor axiom. Now we plug in the theorem of one-point function with $(k, l) = (r - 1, r)$, then

$$\langle h^r, h^r \rangle_d = d^r (-1)^{(d-1)(r+1)} \frac{(-1)^{k-(r+1)}}{d^{k+2}} C_{r}^{k+1}$$

$$= (-1)^{(d-1)(r+1)} \frac{1}{d}.$$ 

Final Reduction

For all \( d \in \mathbb{N} \), \( \sum_{i=1}^{n} l_i = 2r + 1 + (n - 3) \),

\[ \langle h^1, \ldots, h^n \rangle_d = (-1)^{(d-1)(r+1)} N_{l_1, \ldots, l_n} d^{n-3}. \]

For \( n \geq 3 \) and for any 3 markings \( i, j \) and \( k \), \( \psi_j = [D_{ik}|j]^\text{virt} \), the divisor relation can be re-written as

\[
e^*_i L = e^*_j L + \sum_{\beta_1 + \beta_2 = \beta} ((\beta_2 . L)[D_{ik}, \beta_1 |j, \beta_2]^\text{virt} - (\beta_1 . L)[D_{i, \beta_1 |jk, \beta_2}]^\text{virt}).
\]

This leads to

\[
\langle h^{l_1+1}, h^{l_2}, h^{l_3}, \ldots \rangle_{n,d} = \langle h^1, h^{l_2+1}, h^{l_3}, \ldots \rangle_{n,d} + d \langle h^{l_1+l_3}, h^{l_2}, \ldots \rangle_{n-1,d} - d \langle h^1, h^{l_2+l_3}, \ldots \rangle_{n-1,d}.
\]

The desired formula then follows by induction.
Our next task is to compare the genus zero Gromov–Witten invariants of $X$ and $X'$ for curve classes other than the extremal ray.
Degenerations to the Normal Cone to \((X, Z)\)

\[ X = X \times \mathbb{A}^1 \]

\(\Phi : W \to X\) is the blowing-up along \(Z \times \{0\}\)

\(W_t \cong X\) for all \(t \neq 0\)

\(W_0 = Y \cup \tilde{E}\) with \(\tilde{E} = \mathbb{P}_Z(N_Z/X \oplus \mathcal{O})\)

\(\phi = \Phi|_Y : Y \to X\) is the blowing-up along \(Z\)

\(p = \Phi|_{\tilde{E}} : \tilde{E} \to Z \subset X\) is the compactified normal bundle.

\(Y \cap \tilde{E} = E = \mathbb{P}_Z(N_Z/X)\) is the \(\phi\) – exceptional divisor

By similar constructions we also have \(\Phi' : W' \to X' = X' \times \mathbb{A}^1\) and
\(W'_0 = Y' \cup \tilde{E}'\). By definition of ordinary flops we have \(Y = Y'\) and \(E = E'\).
The Key Step (Cohomology Reduction to Local Models)

- \( \alpha \sim (\alpha_1, \alpha_2) \sim (\alpha_1 - e, \alpha_2 + e) \), \( e \in E \)
  \( (\alpha_1 |_E = \alpha_1 |_E, \phi_\ast \alpha_1 + p_\ast \alpha_2 = \alpha) \)

- \( \alpha \sim (\phi \ast \alpha, p \ast \alpha |_{\tilde{Z}}) \sim (\phi \ast \alpha + e, p \ast \alpha |_{\tilde{Z}} - e) \)

\[ \phi \ast g \alpha \]

\[ \alpha_2 \]

- \( \tilde{g} \alpha \sim (\phi^\prime \ast \tilde{g} \alpha, p^\prime \ast \tilde{g} \alpha |_{\tilde{Z}'} ) \)

- \( \alpha_1 = \alpha_1' \Rightarrow \tilde{g} \alpha_2 = \alpha_2' \)

- We can reduce it to the case of projective bundles.
Relative Gromov Witten Invariants

For a smooth pair \((Y, E)\) with \(E \hookrightarrow Y\), \(\Gamma = (g, n, \beta, \rho, \mu)\) with 
\[\mu = (\mu_1, \ldots, \mu_\rho) \in \mathbb{N}^\rho, \quad (\beta, E) = |\mu| := \sum_{i=1}^\rho \mu_i.\]

For \(A \in H^*(Y)^{\otimes n}\) and \(\varepsilon \in H^*(E)^{\otimes \rho}\),

\[\langle A | \varepsilon, \mu \rangle^{(Y, E)}_{\Gamma} := \int_{[\overline{M}_\Gamma(Y, E)]^{\text{virt}}} e_Y^* A \cup e_E^* \varepsilon\]

where \(e_Y : \overline{M}_\Gamma(Y, E) \rightarrow Y^n, \ e_E : \overline{M}_\Gamma(Y, E) \rightarrow E^\rho.\)
Degeneration Formula

\{ e_i \} : a basis of $H^* (E)$; \{ $e^i$ \} : its dual basis.
\{ e_I = e_{i_1} \otimes \cdots \otimes e_{i_\rho} \} : a basis of $H^* (E^\rho)$; \{ $e^I$ \} its dual basis.

The degeneration formula:

$$
\langle \alpha \rangle^X_{g, n, \beta} = \sum_I \sum_{\eta \in \Omega_{\beta}} C_\eta \left. \left\langle \alpha_1 \right| e_I, \mu \right. \bigg| \Gamma_1 \left. \left\langle \alpha_2 \right| e^I, \mu \right. \bigg| \Gamma_2 \right. \bigg( (Y, E) \bigg) \bigg( (\tilde{E}, E) \bigg)
$$

An admissible triple: $\eta = (\Gamma_1, \Gamma_2, I_\rho)$ with $\Gamma_1 + I_\rho \Gamma_2$ connected of type $(g, n, \beta)$.
The constants $C_\eta = m(\mu) / |\text{Aut } \eta|$, where $m(\mu) = \prod \mu_i$ and $\text{Aut } \eta = \{ \sigma \in S_{\rho} \mid \eta^\sigma = \eta \}$. 
First Reduction – Absolute to Relative

To prove $\mathcal{F}\langle \alpha \rangle^X \cong \langle \mathcal{F}\alpha \rangle^{X'}$ (for all $\alpha$), it is enough to show that

$$\mathcal{F}\langle A \mid \varepsilon, \mu \rangle^{(\tilde{E},E)} \cong \langle \mathcal{F}A \mid \varepsilon, \mu \rangle^{(\tilde{E}',E)}$$

for all $A, \varepsilon, \mu$.

For the $n$-point function $\langle \alpha \rangle^X = \sum_{\beta \in NE(X)} \langle \alpha \rangle^X_{\beta} q^\beta$ and $\langle A \mid \varepsilon, \mu \rangle^{\bullet}(\tilde{E},E) := \sum_{\Gamma; \mu_\Gamma = \mu} \frac{1}{|\text{Aut}\Gamma|} \langle A \mid \varepsilon, \mu \rangle_{\Gamma}^{\bullet}(\tilde{E},E) q^\beta_\Gamma$, the degeneration formula gives

$$\langle \alpha \rangle^X = \sum_{\mu} \sum_{l} \sum_{\eta \in \Omega_{\mu}} C_{\eta} \left( \langle \alpha_1 \mid e_l, \mu \rangle_{\Gamma_1}^{\bullet}(Y,E) q^{\beta_1} \right) \left( \langle \alpha_2 \mid e_l, \mu \rangle_{\Gamma_2}^{\bullet}(\tilde{E},E) q^{\beta_2} \right)$$

$$\langle \alpha \rangle^{\bullet X} = \sum_{\mu} m(\mu) \sum_{l} \langle \alpha_1 \mid e_l, \mu \rangle^{\bullet}(Y,E) \langle \alpha_2 \mid e_l, \mu \rangle^{\bullet}(\tilde{E},E).$$
Relative to Absolute

Apply degeneration to the normal cone for $Z \hookrightarrow \tilde{E}$.

$\mathcal{X} = \tilde{E} \times \mathbb{A}^1$

$\Phi : \mathcal{W} \to \mathcal{X}$ is the blowing-up along $Z \times \{0\}$

$W_t \cong \tilde{E}$ for all $t \neq 0$

$W_0 = Y_1 \cup Y_2$

$\pi : Y_1 \cong \mathbb{P}_E(\mathcal{O}_E(-1, -1) \oplus \mathcal{O}) \to E$ a $\mathbb{P}^1$ bundle

$p : Y_2 \cong \tilde{E} \to Z$ the compactified normal bundle.

Denote by $E_0 = E = Y_1 \cap Y_2$ and $E_\infty \cong E$ the zero and infinity divisors of $Y_1$ respectively.
$\alpha_1 = \alpha_1'$

$\mathcal{L} \alpha_2 = \alpha_2'$
Second Reduction

For an ordinary flop $\tilde{E} \rightarrow \tilde{E}'$, to prove

$$\mathcal{F}\langle A \mid \varepsilon, \mu \rangle \cong \langle \mathcal{F}A \mid \varepsilon, \mu \rangle$$

for any $A$ and $(\varepsilon, \mu)$, it is enough to show that

$$\mathcal{F}\langle A, \tau_{k_1}\varepsilon_1, \ldots, \tau_{k_\rho}\varepsilon_\rho \rangle \tilde{E} \cong \langle \mathcal{F}A, \tau_{k_1}\varepsilon_1, \ldots, \tau_{k_\rho}\varepsilon_\rho \rangle \tilde{E}'$$

for any possible insertions $A \in H^*(\tilde{E})^{\oplus n}$, $k_j \in \mathbb{N} \cup \{0\}$ and $\varepsilon_j \in H^*(E)$. 
Key Idea

Given \(\langle \alpha_1, \ldots, \alpha_n | \varepsilon, \mu \rangle\) on \((\tilde{E}, E)\), by using induction on the triple \((|\mu|, n, \rho)\) in the lexicographical order with \(\rho\) in the reverse order, we analyze the degeneration formula

\[
\langle \alpha_1, \ldots, \alpha_n, \tau_{\mu_1 - 1}e_1, \ldots, \tau_{\mu_\rho - 1}e_\rho | e_{l''}, \mu' \rangle (Y_{1,E}) \langle \alpha_1, \ldots, \alpha_n | e_{l''}, \mu' \rangle (\tilde{E}, E) + R,
\]

where \(R\) denotes the remaining terms which either have smaller total contact order or have number of insertions fewer than \(n\) on the \((\tilde{E}, E)\) side or the invariants on \((\tilde{E}, E)\) are disconnected ones.

The highest order term in the sum consists of the single term

\[
C(\mu)\langle \alpha_1, \ldots, \alpha_n | \varepsilon, \mu \rangle (\tilde{E}, E)\text{ where } C(\mu) \neq 0.
\]
The Final Job

Let $\langle \alpha \rangle = \langle \alpha_1, \ldots, \alpha_n \rangle$ with $\alpha_i \in H^*(\tilde{E}) \cup \tau \cdot H^*(E)$. If $d_2 \neq 0$ then

$$\mathcal{F}\langle \alpha \rangle \cong \langle \mathcal{F}\alpha \rangle.$$  

For example, descendent invariants for simple $\mathbb{P}^2$ flop with $n = 3$. Let $q_1 = q^\ell$, $q_2 = q^\gamma$, then $\mathcal{F}q^\ell = q^{-\ell'} = q'_{1}^{-1}$, $\mathcal{F}q^\gamma = q^{\ell'+\gamma'} = q'_1 q'_2$.

$$\langle h^2, h^2, \tau_4 \xi \rangle = 3q_1 q_2 - 6 \frac{q_1 q_2}{1 + q_1}$$

$$\mathcal{F}\langle h^2, h^2, \tau_4 \xi \rangle = 3q_{1}^{-1} q'_{1} q'_{2} - 6 \frac{q_{1}^{-1} q'_{1} q'_{2}}{1 + q'_{1}^{-1}} = 3q'_{2} - 6 \frac{q'_{1} q'_{2}}{1 + q'_{1}}$$

$$\langle \mathcal{F}h^2, \mathcal{F}h^2, \mathcal{F}\tau_4 \xi \rangle = \langle (\xi' - h')^2, (\xi' - h')^2, \tau_4 \xi' \rangle = 3q'_{2} - 6 \frac{q'_{1} q'_{2}}{1 + q'_{1}}$$
Sketch of Proof

Let \(\beta \in NE(\tilde{E}) = i_*NE(Z) \oplus \mathbb{Z}_+\gamma\) with \(\gamma\) the fiber line class of \(\tilde{E} \to Z\), say \(\beta = d_1\ell + d_2\gamma\) (here \(d_2 \neq 0\)).

The virtual dimension of \(n\) point invariants is \((r + 2)d_2 + 2r + n - 2\).

For a fixed cohomology insertions, there could be at most one \(d_2\) supporting non-trivial invariant.

By reordering, \(\alpha_n = \tau_s\xi a, \ s \geq 0\). Write \(\alpha_1 = \tau_k h^l\xi^j\).

We use induction on \((d_2, n, k, l, j)\).

The induction procedure is to move divisors in \(\alpha_1\) into \(\alpha_n\) in the order of \(\psi, h\) and \(\xi\) and based on two initial cases:

- \(d_2 = 0\) (back to the extremal case)
- (One point descendant invariant)

\[
\mathcal{F} \langle \tau_k \xi.\alpha \rangle^X = \langle \tau_k \mathcal{F}(\xi.\alpha) \rangle^{X'} = \langle \tau_k \xi'.\mathcal{F}\alpha \rangle^{X'}.
\]