On a Local-Global Property of Algebraic Dynamics

Liang-Chung Hsia
Department of Mathematics
National Central University
Taiwan, R.O.C.

East Asia Number Theory Conference at KAIST
January 22, 2008
Outline

1. A Local-Global Principle in Number Theory
   - The Hasse Principle
   - The Brauer-Manin Obstruction

2. A Dynamical Analogue
   - Algebraic Dynamics
   - Statement of the Main Results
   - Sketch of the Proof

3. Related Questions
### Notations.

- $K$: a number field.
- $M_K$: the set of inequivalent places of $K$.
- $M_K^\infty$: the set of archimedean places of $K$.
- $p_v$: the prime ideal associated to a finite place $v \in M_K$.
- $K_v$: the completion of $K$ with respect to $v \in M_K$.
- $A_K$: the ring of adèles of $K$.
- $X/K$: an projective variety defined over $K$.
- $X(F)$: the set of points of $X$ defined over the field $F$. 
From global to local

We have

\[ X(K) \hookrightarrow X(K_{v}) \quad \text{for all } v \in M_{K}. \]

Suppose \( X(K) \neq \emptyset \), then

\[ X(K_{v}) \neq \emptyset \quad \forall v \in M_{K} \quad (\text{The Hasse condition}). \]

**Remark**

It's always true that \( X(K_{v}) \neq \emptyset \) for all but finitely many \( v \in M_{K} \). So, one only has to check whether or not \( X(K_{v}) \neq \emptyset \) for the remaining finite subset of \( M_{K} \) and in many cases this can be done in a finite steps of computations.
From global to local

We have

\[ X(K) \hookrightarrow X(K_v) \quad \text{for all } v \in M_K. \]

Suppose \( X(K) \neq \emptyset \), then

\[ X(K_v) \neq \emptyset \quad \forall v \in M_K \quad (\text{The Hasse condition}). \]

Remark

It’s always true that \( X(K_v) \neq \emptyset \) for all but finitely many \( v \in M_K \). So, one only has to check whether or not \( X(K_v) \neq \emptyset \) for the remaining finite subset of \( M_K \) and in many cases this can be done in a finite steps of computations.
From global to local

We have

\[ X(K) \hookrightarrow X(K_v) \quad \text{for all } v \in M_K. \]

Suppose \( X(K) \neq \emptyset \), then

\[ X(K_v) \neq \emptyset \quad \forall v \in M_K \] (The Hasse condition).

Remark

It’s always true that \( X(K_v) \neq \emptyset \) for all but finitely many \( v \in M_K \).
So, one only has to check whether or not \( X(K_v) \neq \emptyset \) for the remaining finite subset of \( M_K \) and in many cases this can be done in a finite steps of computations.
We have

\[ X(K) \leftrightarrow X(K_v) \quad \text{for all } v \in M_K. \]

Suppose \( X(K) \neq \emptyset \), then

\[ X(K_v) \neq \emptyset \quad \forall v \in M_K \quad (\text{The Hasse condition}). \]

**Remark**

It’s always true that \( X(K_v) \neq \emptyset \) for all but finitely many \( v \in M_K \). So, one only has to check whether or not \( X(K_v) \neq \emptyset \) for the remaining finite subset of \( M_K \) and in many cases this can be done in a finite steps of computations.
From local to global

**Question**

Is the Hasse condition a sufficient condition to imply \( X(K) \neq \emptyset \)?

If the Hasse condition is a sufficient condition for \( X(K) \neq \emptyset \) then \( X/K \) is said to satisfy the *Hasse principle*.

**Example (The Hasse-Minkowski Theorem)**

\( F(x_1, \ldots, x_n) \): a quadratic form defined over \( K \) where \( n \geq 2 \).

\[
F(x_1, \ldots, x_n) = 0 \quad \text{has nontrivial solution } (x_1, \ldots, x_n) \in K^n
\]

\[
\uparrow
\]

\[
F(x_1, \ldots, x_n) = 0 \quad \text{has nontrivial solution } (x_1, \ldots, x_n) \in K^n_v
\]

for all \( v \in M_K \).
From local to global

Question

Is the Hasse condition a sufficient condition to imply $X(K) \neq \emptyset$?

If the Hasse condition is a sufficient condition for $X(K) \neq \emptyset$ then $X/K$ is said to satisfy the Hasse principle.

Example (The Hasse-Minkowski Theorem)

$F(x_1, \ldots, x_n) : a$ quadratic form defined over $K$ where $n \geq 2$.

$F(x_1, \ldots, x_n) = 0$ has nontrivial solution $(x_1, \ldots, x_n) \in K^n$

$\Updownarrow$

$F(x_1, \ldots, x_n) = 0$ has nontrivial solution $(x_1, \ldots, x_n) \in K_v^n$ for all $v \in M_K$. 
Question

Is the Hasse condition a sufficient condition to imply $X(K) \neq \emptyset$?

If the Hasse condition is a sufficient condition for $X(K) \neq \emptyset$ then $X/K$ is said to satisfy the *Hasse principle*.

Example (The Hasse-Minkowski Theorem)

$F(x_1, \ldots, x_n)$: a quadratic form defined over $K$ where $n \geq 2$.

$F(x_1, \ldots, x_n) = 0$ has nontrivial solution $(x_1, \ldots, x_n) \in K^n$

$\iff$

$F(x_1, \ldots, x_n) = 0$ has nontrivial solution $(x_1, \ldots, x_n) \in K_v^n$ for all $v \in M_K$. 
Question

Is the Hasse condition a sufficient condition to imply $X(K) \neq \emptyset$?

If the Hasse condition is a sufficient condition for $X(K) \neq \emptyset$ then $X/K$ is said to satisfy the Hasse principle.

Example (The Hasse-Minkowski Theorem)

$F(x_1, \ldots, x_n)$: a quadratic form defined over $K$ where $n \geq 2$.

$F(x_1, \ldots, x_n) = 0$ has nontrivial solution $(x_1, \ldots, x_n) \in K^n$

$\Leftrightarrow$

$F(x_1, \ldots, x_n) = 0$ has nontrivial solution $(x_1, \ldots, x_n) \in K_v^n$ for all $v \in M_K$. 
### Counter-example to the Hasse principle

**Example (C.-E. Lind)**

Let $X/\mathbb{Q}$ be the curve defined by equation

$$2y^2 = x^4 - 17.$$ 

Then,

$$X(\mathbb{Q}_v) \neq \emptyset \quad \forall v \in M_{\mathbb{Q}} \quad \text{but} \quad X(\mathbb{Q}) = \emptyset.$$

**Example (S. Selmer)**

Let $X/\mathbb{Q}$ be the cubic curve defined by equation

$$3X^3 + 4Y^3 + 5Z^3 = 0.$$ 

Then,

$$X(\mathbb{Q}_v) \neq \emptyset \quad \forall v \in M_{\mathbb{Q}} \quad \text{but} \quad X(\mathbb{Q}) = \emptyset.$$
### Example (C.-E. Lind)

Let \( X/\mathbb{Q} \) be the curve defined by equation

\[
2y^2 = x^4 - 17.
\]

Then,

\[
X(\mathbb{Q}_v) \neq \emptyset \quad \forall v \in M_{\mathbb{Q}} \quad \text{but} \quad X(\mathbb{Q}) = \emptyset.
\]

### Example (S. Selmer)

Let \( X/\mathbb{Q} \) be the cubic curve defined by equation

\[
3X^3 + 4Y^3 + 5Z^3 = 0.
\]

Then,

\[
X(\mathbb{Q}_v) \neq \emptyset \quad \forall v \in M_{\mathbb{Q}} \quad \text{but} \quad X(\mathbb{Q}) = \emptyset.
\]
Counter-example to the Hasse principle

Example (C.-E. Lind)
Let \( X / \mathbb{Q} \) be the curve defined by equation
\[
2y^2 = x^4 - 17.
\]
Then,
\[
X(\mathbb{Q}_v) \neq \emptyset \quad \forall v \in M_{\mathbb{Q}} \quad \text{but} \quad X(\mathbb{Q}) = \emptyset.
\]

Example (S. Selmer)
Let \( X / \mathbb{Q} \) be the cubic curve defined by equation
\[
3X^3 + 4Y^3 + 5Z^3 = 0.
\]
Then,
\[
X(\mathbb{Q}_v) \neq \emptyset \quad \forall v \in M_{\mathbb{Q}} \quad \text{but} \quad X(\mathbb{Q}) = \emptyset.
\]
Putting local information together

Question
What is the obstruction for $X$ to satisfy the Hasse principle?

$X(A_K) := \prod_{v \in M_K} X(K_v)$ (the set of adèlic points of $X$).

- Since $X(K_v)$ is compact with respect to the topology induced by that of $K_v$, it follows that $X(A_K)$ is a compact topological space with respect to the product of topologies on $X(K_v)$ for all $v \in M_K$.
- We have

$$X(K) \hookrightarrow X(A_K)$$

$$P \mapsto (P)_v \in M_K$$
Putting local information together

Question

What is the obstruction for $X$ to satisfy the Hasse principle?

$$X(A_K) := \prod_{v \in M_K} X(K_v) \quad (\text{the set of adèlic points of } X).$$

- Since $X(K_v)$ is compact with respect to the topology induced by that of $K_v$, it follows that $X(A_K)$ is a compact topological space with respect to the product of topologies on $X(K_v)$ for all $v \in M_K$.
- We have

$$X(K) \hookrightarrow X(A_K)$$

$$P \mapsto (P)_v \in M_K$$
Putting local information together

Question

What is the obstruction for $X$ to satisfy the Hasse principle?

$$X(A_K) := \prod_{v \in M_K} X(K_v) \quad \text{(the set of adèlic points of $X$)}.$$  

- Since $X(K_v)$ is compact with respect to the topology induced by that of $K_v$, it follows that $X(A_K)$ is a compact topological space with respect to the product of topologies on $X(K_v)$ for all $v \in M_K$.

- We have

$$X(K) \hookrightarrow X(A_K)$$

$$P \mapsto (P)_{v \in M_K}$$
Question

What is the obstruction for $X$ to satisfy the Hasse principle?

$$X(A_K) := \prod_{v \in M_K} X(K_v) \quad \text{(the set of adèlic points of } X).$$

- Since $X(K_v)$ is compact with respect to the topology induced by that of $K_v$, it follows that $X(A_K)$ is a compact topological space with respect to the product of topologies on $X(K_v)$ for all $v \in M_K$.

- We have

$$X(K) \hookrightarrow X(A_K)$$

$$P \mapsto (P)_{v \in M_K}$$
Putting local information together

Question

What is the obstruction for $X$ to satisfy the Hasse principle?

$$X(A_K) := \prod_{v \in M_K} X(K_v)$$ (the set of adèlic points of $X$).

- Since $X(K_v)$ is compact with respect to the topology induced by that of $K_v$, it follows that $X(A_K)$ is a compact topological space with respect to the product of topologies on $X(K_v)$ for all $v \in M_K$.
- We have

$$X(K) \hookrightarrow X(A_K)$$
$$P \mapsto (P)_{v \in M_K}$$
The Brauer set

Question

How to cut out the global information $X(K)$ from $X(A_K)$?

Remark (Y. Manin)

There is a closed subset $X(A_K)^{Br}$ of $X(A_K)$ defined by certain conditions related to the Brauer group $\text{Br}(X)$ such that

$$X(K) \subset X(A_K)^{Br} \subset X(A_K).$$

Theorem (Y. Manin)

$X/K$ : projective, smooth curve with $g(X) = 1$, $E = \text{Jac}(X)$. Suppose that the Tate-Shafarevich group $\text{III}(E)$ is finite. If $X(K) = \emptyset$ and $X(A_K) \neq \emptyset$, then $X(A_K)^{Br} = \emptyset$. 
The Brauer set

Question

How to cut out the global information $X(K)$ from $X(A_K)$?

Remark (Y. Manin)

There is a closed subset $X(A_K)^{Br}$ of $X(A_K)$ defined by certain conditions related to the Brauer group $Br(X)$ such that

$$X(K) \subset X(A_K)^{Br} \subset X(A_K).$$

Theorem (Y. Manin)

$X/K$: projective, smooth curve with $g(X) = 1$. $\mathcal{E} = \text{Jac}(X)$. Suppose that the Tate-Shafarevich group $\Sha(\mathcal{E})$ is finite. If $X(K) = \emptyset$ and $X(A_K) \neq \emptyset$, then $X(A_K)^{Br} = \emptyset$. 
The Brauer set

Question
How to cut out the global information $X(K)$ from $X(A_K)$?

Remark (Y. Manin)
There is a closed subset $X(A_K)^{Br}$ of $X(A_K)$ defined by certain conditions related to the Brauer group $\text{Br}(X)$ such that

$$X(K) \subset X(A_K)^{Br} \subset X(A_K).$$

Theorem (Y. Manin)
$X/K$ : projective, smooth curve with $g(X) = 1$. $E = \text{Jac}(X)$.
Suppose that the Tate-Shafarevich group $\text{III}(E)$ is finite. If $X(K) = \emptyset$ and $X(A_K) \neq \emptyset$, then $X(A_K)^{Br} = \emptyset$. 
The Brauer set

**Question**
How to cut out the global information $X(K)$ from $X(A_K)$?

**Remark (Y. Manin)**
There is a closed subset $X(A_K)^{Br}$ of $X(A_K)$ defined by certain conditions related to the Brauer group $\text{Br}(X)$ such that

$$X(K) \subset X(A_K)^{Br} \subset X(A_K).$$

**Theorem (Y. Manin)**

$X/K :$ projective, smooth curve with $g(X) = 1$, $\mathcal{E} = \text{Jac}(X)$.

Suppose that the Tate-Shafarevich group $\text{III}(\mathcal{E})$ is finite. If $X(K) = \emptyset$ and $X(A_K) \neq \emptyset$, then $X(A_K)^{Br} = \emptyset$. 
Capturing the Brauer set

Let $X/K$ be a smooth, projective curve defined over $K$. Let $A = \text{Jac}(X) = \text{Pic}^0(X)$, the Jacobian of the curve $X$ and a fixed jacobian embedding

$$j : X \hookrightarrow A$$

which assumed to be defined over $K$.

- $A(K_v)^0$ the identity component of $A(K_v)$ for $v \in M^\infty_K$ and 0 for $v \in M^0_K$.
- $A(A_K)_\bullet = \prod_{v \in M_K} A(K_v)/A(K_v)^0$.
- $X(A_K)_\bullet$ the image of $j(X(A_K))$ under the projection $A(A_K) \to A(A_K)_\bullet$. 

Liang-Chung Hsia  
Department of Mathematics  
National Cent
Let $X/K$ be a smooth, projective curve defined over $K$.

Let $A = \text{Jac}(X)(= \text{Pic}^0(X))$, the Jacobian of the curve $X$ and a fixed jacobian embedding

$$j : X \hookrightarrow A$$

which assumed to be defined over $K$.

- $A(K_v)^0$ the identity component of $A(K_v)$ for $v \in M_K^\infty$ and 0 for $v \in M_K^0$.
- $A(A_K)_0 = \prod_{v \in M_K} A(K_v)/A(K_v)^0$.
- $X(A_K)_0$ the image of $j(X(A_K))$ under the projection $A(A_K) \rightarrow A(A_K)_0$. 

Liang-Chung Hsia  Department of Mathematics  National Cent
Let $X/K$ be a smooth, projective curve defined over $K$. Let $A = \text{Jac}(X) (= \text{Pic}^0(X))$, the Jacobian of the curve $X$ and a fixed jacobian embedding

$$j : X \hookrightarrow A$$

which assumed to be defined over $K$.

- $A(K_v)^0$ the identity component of $A(K_v)$ for $v \in M_K^\infty$ and 0 for $v \in M_K^0$.
- $A(A_K)_\bullet = \prod_{v \in M_K} A(K_v)/A(K_v)^0$.
- $X(A_K)_\bullet$ the image of $j(X(A_K))$ under the projection $A(A_K) \twoheadrightarrow A(A_K)_\bullet$. 

Liang-Chung Hsia  Department of Mathematics  National Cent
Capturing the Brauer set

Let $X/K$ be a smooth, projective curve defined over $K$. Let $A = \text{Jac}(X)(= \text{Pic}^0(X))$, the Jacobian of the curve $X$ and a fixed jacobian embedding

$$j : X \hookrightarrow A$$

which assumed to be defined over $K$.

$A(K_v)^0$ the identity component of $A(K_v)$ for $v \in M_K^\infty$ and $0$ for $v \in M_K^0$.

$A(A_k) \cdot = \prod_{v \in M_K} A(K_v)/A(K_v)^0$.

$X(A_k) \cdot$ the image of $j(X(A_K))$ under the projection $A(A_K) \twoheadrightarrow A(A_k) \cdot$. 

Adèlic closure of Mordell-Weil Groups

\[ \overline{A(K)} : \text{the topological closure of } A(K) \text{ inside } A(A_k). \]

**Theorem**

(V. Sharaschkin) Suppose that \( \mathbb{III}(A) \) is finite. Then,

\[ X(K) \simeq X(A_k) \cap A(K) \subseteq X(A_k) \cap \overline{A(K)} \simeq X(A_K)^{Br}. \]
Adèlic closure of Mordell-Weil Groups

\[ \overline{A(K)} : \text{the topological closure of } A(K) \text{ inside } A(A_k). \]

**Theorem**

(V. Sharaschkin) Suppose that \( \mathfrak{I}(A) \) is finite. Then,

\[ X(K) \cong X(A_k) \bullet \cap A(K) \subseteq X(A_k) \bullet \cap \overline{A(K)} \cong X(A_K)^{Br}. \]
A Local-Global Principle in Number Theory
A Dynamical Analogue
Related Questions

The Hasse Principle
The Brauer-Manin Obstruction

Function field analogue.

$F$ a finitely generated extension and of transcendence degree 1 over a field $k$,

$A$ an abelian variety over $F$,

$X$ a closed $F$-subscheme of $A$.

$X(A_k)_\bullet$, $A(A_k)_\bullet$ defined as above.

Theorem
(B. Poonen and F. Voloch)

(1) If $\text{Char } k = 0$ then $X(F) = X(A) \cap \overline{A(F)}$.

(2) If $\text{Char } k = p > 0$ then, under certain mild conditions,

$X(F) = X(A_k)_\bullet \cap \overline{A(F)}$. 
Function field analogue.

\( F \) a finitely generated extension and of transcendence degree 1 over a field \( k \),
\( A \) an abelian variety over \( F \),
\( X \) a closed \( F \)-subscheme of \( A \).
\( X(A_k) \bullet, A(A_k) \bullet \) defined as above.

**Theorem**

(B. Poonen and F. Voloch)

1. If \( \text{Char } k = 0 \) then \( X(F) = X(A) \cap \overline{A(F)} \).
2. If \( \text{Char } k = p > 0 \) then, under certain mild conditions, 
\( X(F) = X(A_k) \bullet \cap \overline{A(F)} \).
Orbits, (pre)periodic points

Let $X$ be any algebraic variety over $K$ and let

$$\varphi : X \to X \text{ of degree } d \geq 2 \text{ over } K.$$  

Dynamical system associated to $\varphi$ on $X$ is roughly the classification of points in $X$ under the action of $\varphi$ via iterations $\{\varphi^n | n \geq 0\}$

Let $P \in X(K)$. The (forward) orbit of $P$

$$O_\varphi(P) := \{\varphi^n(P) | n = 0, 1, 2, \ldots\}.$$  

$P$ is called preperiodic if $\#O_\varphi(P) < \infty$; wandering if $\#O_\varphi(P) = \infty$.

Comparison: dynamical systems v.s. abelian variety

preperiodic points $\longleftrightarrow$ torsion points

wandering points $\longleftrightarrow$ point of infinite order

orbit $O_\varphi(P) \longleftrightarrow$ Cyclic subgroup of $A(K)$.

Canonical height $\hat{h}_\varphi \longleftrightarrow$ Néron-Tate height on $A$
Orbits, (pre)periodic points

Let $X$ be any algebraic variety over $K$ and let
\[ \varphi : X \to X \] of degree $d \geq 2$ over $K$.

Dynamical system associated to $\varphi$ on $X$ is roughly the classification of points in $X$ under the action of $\varphi$ via iterations $\{ \varphi^n \mid n \geq 0 \}$.

Let $P \in X(K)$. The (forward) orbit of $P$
\[ \mathcal{O}_\varphi(P) := \{ \varphi^n(P) \mid n = 0, 1, 2, \ldots \}. \]

$P$ is called \textit{preperiodic} if $\# \mathcal{O}_\varphi(P) < \infty$; \textit{wandering} if $\# \mathcal{O}_\varphi(P) = \infty$.

Comparison: dynamical systems v.s. abelian variety

- preperiodic points $\leftrightarrow$ torsion points
- wandering points $\leftrightarrow$ point of infinite order
- orbit $\mathcal{O}_\varphi(P)$ $\leftrightarrow$ Cyclic subgroup of $A(K)$.
- Canonical height $\hat{h}_\varphi$ $\leftrightarrow$ Néron-Tate height on $A$. 

Liang-Chung Hsia  Department of Mathematics National Central University
Taiwan, R.O.C.

On a Local-Global Property of Algebraic Dynamics
Orbits, (pre)periodic points

Let $X$ be any algebraic variety over $K$ and let

$$\varphi : X \to X \text{ of degree } d \geq 2 \text{ over } K.$$ 

Dynamical system associated to $\varphi$ on $X$ is roughly the classification of points in $X$ under the action of $\varphi$ via iterations $\{\varphi^n \mid n \geq 0\}$

Let $P \in X(K)$. The (forward) orbit of $P$

$$O_\varphi(P) := \{\varphi^n(P) \mid n = 0, 1, 2, \ldots\}.$$ 

$P$ is called **preperiodic** if $\#O_\varphi(P) < \infty$; **wandering** if $\#O_\varphi(P) = \infty$.

**Comparison:** dynamical systems v.s. abelian variety

- preperiodic points $\longleftrightarrow$ torsion points
- wandering points $\longleftrightarrow$ point of infinite order
- orbit $O_\varphi(P) \longleftrightarrow$ Cyclic subgroup of $A(K)$.
- Canonical height $\hat{h}_\varphi \longleftrightarrow$ Néron-Tate height on $A$
Let $X$ be any algebraic variety over $K$ and let

$$\varphi : X \to X$$

of degree $d \geq 2$ over $K$.

Dynamical system associated to $\varphi$ on $X$ is roughly the classification of points in $X$ under the action of $\varphi$ via iterations $\{\varphi^n \mid n \geq 0\}$

Let $P \in X(K)$. The (forward) orbit of $P$

$$\mathcal{O}_\varphi(P) := \{\varphi^n(P) \mid n = 0, 1, 2, \ldots\}.$$ 

$P$ is called preperiodic if $\#\mathcal{O}_\varphi(P) < \infty$; wandering if $\#\mathcal{O}_\varphi(P) = \infty$.

Comparison: dynamical systems v.s. abelian variety

preperiodic points $\longleftrightarrow$ torsion points

wandering points $\longleftrightarrow$ point of infinite order

orbit $\mathcal{O}_\varphi(P)$ $\longleftrightarrow$ Cyclic subgroup of $A(K)$.

Canonical height $\hat{h}_\varphi$ $\longleftrightarrow$ Néron-Tate height on $A$. 
Orbits, (pre)periodic points

Let \( X \) be any algebraic variety over \( K \) and let

\[ \varphi : X \rightarrow X \] of degree \( d \geq 2 \) over \( K \).

Dynamical system associated to \( \varphi \) on \( X \) is roughly the classification of points in \( X \) under the action of \( \varphi \) via iterations \( \{ \varphi^n \mid n \geq 0 \} \)

Let \( P \in X(K) \). The (forward) orbit of \( P \)

\[ O_\varphi(P) := \{ \varphi^n(P) \mid n = 0, 1, 2, \ldots \}. \]

\( P \) is called preperiodic if \( \#O_\varphi(P) < \infty \); wandering if \( \#O_\varphi(P) = \infty \).

Comparison: dynamical systems v.s. abelian variety

preperiodic points \( \leftrightarrow \) torsion points
wandering points \( \leftrightarrow \) point of infinite order
orbit \( O_\varphi(P) \) \( \leftrightarrow \) Cyclic subgroup of \( A(K) \).

Canonical height \( \hat{h}_\varphi \) \( \leftrightarrow \) Néron-Tate height on \( A \)
A local-global question

Let $V$ be a subvariety of $X$ and let $P \in X(K)$ be wandering.

$\overline{O_\varphi(P)} = \text{the topological closure of } O_\varphi(P) \text{ in } X(A_K)$.

We have an inclusion

$$O_\varphi(P) \cap V(K) \subseteq \overline{O_\varphi(P)} \cap V(A_K).$$

Question

Is it true that

$$O_\varphi(P) \cap V(K) = \overline{O_\varphi(P)} \cap V(A_K)?$$

Or, when does the equality hold?
A local-global question

Let $V$ be a subvariety of $X$ and let $P \in X(K)$ be wandering. 

$$\overline{O_\varphi(P)} = \text{the topological closure of } O_\varphi(P) \text{ in } X(A_K).$$

We have an inclusion

$$O_\varphi(P) \cap V(K) \subseteq \overline{O_\varphi(P)} \cap V(A_K).$$

Question

Is it true that 

$$O_\varphi(P) \cap V(K) = \overline{O_\varphi(P)} \cap V(A_K)?$$

Or, when does the equality hold?
A subvariety $W$ of $X$ is $\varphi$-preperiodic if there are integers $n > m$ such that $\varphi^n(W) = \varphi^m(W)$. If also $\dim(W) \geq 1$, we say that $W$ is nontrivial.

Let $V^{\text{pp}}_\varphi$ be the union of all nontrivial $\varphi$-preperiodic subvarieties of $V$. Then we say that $V(K)$ is Brauer–Manin unobstructed (for $\varphi$) if for every point $P \in X(K)$ satisfying $\mathcal{O}_\varphi(P) \cap V^{\text{pp}}_\varphi = \emptyset$ we have

$$\mathcal{O}_\varphi(P) \cap V(K) = \overline{\mathcal{O}_\varphi(P) \cap V(A_K)}.$$
Dynamical Brauer-Manin

**Definition**

A subvariety $W$ of $X$ is $\varphi$-preperiodic if there are integers $n > m$ such that $\varphi^n(W) = \varphi^m(W)$. If also $\dim(W) \geq 1$, we say that $W$ is nontrivial.

**Definition**

Let $V^{pp}_\varphi$ be the union of all nontrivial $\varphi$-preperiodic subvarieties of $V$. Then we say that $V(K)$ is Brauer–Manin unobstructed (for $\varphi$) if for every point $P \in X(K)$ satisfying $O_\varphi(P) \cap V^{pp}_\varphi = \emptyset$ we have

$$O_\varphi(P) \cap V(K) = \overline{O_\varphi(P)} \cap V(A_K).$$
Power maps and translated tori

Theorem (L.C.Hsia and J. Silverman)

Let
\[ \varphi : \mathbb{P}^2 \to \mathbb{P}^2, \quad \varphi([X, Y, Z]) = [X^d, Y^d, Z^d] \]
for some \( d \geq 2 \). Let \( k, \ell \geq 1 \), let \( A, B \in K \), and let \( V \subset \mathbb{P}^2 \) be the curve
\[ V : AX^k = BY^\ell. \]

Then, \( V(K) \) is Brauer-Manin unobstructed. That is, for a wandering point \( P \in \mathbb{P}^2(K) \), one of the following two statements is true:

(i) \( \mathcal{O}_{\varphi}(P) \cap V(K) = \overline{\mathcal{O}_{\varphi}(P)} \cap V(A_K) \).

(ii) The variety \( V \) is preperiodic for \( \varphi \), and if \( \mathcal{O}_{\varphi}(P) \cap V(K) \) is non-empty then there exists an \( i \geq 0 \) such that \( \mathcal{O}_{\varphi}(P) \cap \varphi^i(V) \) is an infinite set.
Theorem (L.C.Hsia and J. Silverman)

Let $A/K$ be an abelian variety, and let $B/K$ be an abelian subvariety of $A$ of codimension $1$. Fix a $T \in A(K)$ and let $V = B + T$ be the translation of $B$ by $T$. Let $d \geq 2$ and consider the multiplication-by-$d$ map

$$[d] : A \to A.$$ 

Then, $V(K)$ is Brauer-Manin unobstructed. That is, for nontorsion point $P \in A(K)$, one of the following two statements is true:

(i) $O_d(P) \cap V(K) = \overline{O_d(P)} \cap V(A_K)$.

(ii) The variety $V$ is $[d]$-preperiodic.

Further, in case (ii), the point $T$ has finite order in the quotient variety $A/B$. 

Translated abelian subvarieties
Theorem (L.C. Hsia and J. Silverman)

Let $A/K$ be an abelian variety and let $V/K$ be a subvariety of $A$. Assume that

(i) $V$ does not contain a translate of a positive-dimensional abelian subvariety of $A$.

(ii) $V(K) = V(A_k)_\bullet \cap \overline{A(K)}$, where the closure and the equality take place in $A(A_k)_\bullet$.

Then, $V(K)$ is Brauer-Manin unobstructed for any $\varphi \in \text{End}(A)$ such that $\deg(\varphi) \geq 2$ and that $\mathbb{Z}[\varphi]$ is an integral domain.
The case of power maps

Recall the $d^{th}$-power map

$$\varphi([X, Y, Z]) = [X^d, Y^d, Z^d]$$

and $V$ is the curve defined by equation

$$AX^k = BY^\ell.$$  

Let $P = [\alpha, \beta, \gamma] \in \mathbb{P}^2(K)$ be a wandering point for $\varphi$.

(The main case) Consider the case $\alpha \beta \gamma \neq 0$ and $AB \neq 0$

Suppose there exists

$$Q = (Q_v)_{v \in M_K} \in \overline{\mathcal{O}_\varphi(P)} \cap V(A_K) \setminus \mathcal{O}_\varphi(P) \cap V(K).$$

We will show that $B/A$ is a root of unity and hence prove that $V$ is actually preperiodic for the power map $\varphi$. 
The case of power maps

Recall the $d^{th}$-power map

$$\varphi([X, Y, Z]) = [X^d, Y^d, Z^d]$$

and $V$ is the curve defined by equation

$$AX^k = BY^\ell.$$

Let $P = [\alpha, \beta, \gamma] \in \mathbb{P}^2(K)$ be a wandering point for $\varphi$.

(The main case) Consider the case $\alpha \beta \gamma \neq 0$ and $AB \neq 0$

Suppose there exists

$$Q = (Q_v)_{v \in M_K} \in \overline{\mathcal{O}_\varphi(P)} \cap V(A_K) \setminus \mathcal{O}_\varphi(P) \cap V(K).$$

We will show that $B/A$ is a root of unity and hence prove that $V$ is actually preperiodic for the power map $\varphi$. 
The case of power maps

Recall the $d^{\text{th}}$-power map

$$\varphi([X, Y, Z]) = [X^d, Y^d, Z^d]$$

and $V$ is the curve defined by equation

$$AX^k = BY^\ell.$$

Let $P = [\alpha, \beta, \gamma] \in \mathbb{P}^2(K)$ be a wandering point for $\varphi$.

(The main case) Consider the case $\alpha \beta \gamma \neq 0$ and $AB \neq 0$

Suppose there exists

$$Q = (Q_v)_{v \in M_K} \in \overline{\mathcal{O}_{\varphi}(P)} \cap V(A_K) \setminus \mathcal{O}_{\varphi}(P) \cap V(K).$$

We will show that $B/A$ is a root of unity and hence prove that $V$ is actually preperiodic for the power map $\varphi$. 
Adèlic closure of an orbit

Let

\[ Q = (Q_v)_{v \in M_K} \in \prod_{v \in M_K} \mathbb{P}^2(K_v). \]

Then,

\[ Q \in \overline{\mathcal{O}_\varphi(P)} \setminus \mathcal{O}_\varphi(P) \]

if and only if there is an infinite set of positive integers \( N_{P,Q} \subset \mathbb{N} \) such that

\[ Q_v = \nu\text{-}\lim_{n \to \infty} \varphi^n(P) \quad \forall \ v \in M_K. \quad (1) \]

- “\( \nu\text{-}\lim \)” indicates the limit is being taken in the \( \nu \)-adic topology.
- \( N_{P,Q} \) depends only on \( P \) and \( Q \), but it must be independent of \( v \in M_K \).

Liang-Chung Hsia  Department of Mathematics  National Cent
Adèlic closure of an orbit

Let

\[ Q = (Q_v)_{v \in M_K} \in \prod_{v \in M_K} \mathbb{P}^2(K_v). \]

Then,

\[ Q \in \overline{O_\varphi(P)} \setminus O_\varphi(P) \]

if and only if there is an infinite set of positive integers \( \mathcal{N}_{P,Q} \subset \mathbb{N} \) such that

\[ Q_v = \nu-lim\ \varphi^n(P) \quad \forall \ v \in M_K. \quad (1) \]

- “\( \nu \)-lim” indicates the limit is being taken in the \( \nu \)-adic topology.
- \( \mathcal{N}_{P,Q} \) depends only on \( P \) and \( Q \), but it must be independent of \( v \in M_K \).
Dehomogenize so that $P = [\alpha, \beta, 1]$ and let

$$S = M_K^\infty \cup \{ v \in M_K \mid |\alpha|_v \neq 1 \text{ or } |\beta|_v \neq 1 \}$$

For $v \notin S$, we have

$$Q_v = \nu\lim_{n \in \mathbb{N}, P, Q} \varphi^n(P) = [x_v, y_v, 1] \in V(K_v)$$

It follows that

$$\nu\lim_{n \in \mathbb{N}, P, Q} \left( \frac{\alpha^k}{\beta^\ell} \right)^d = \frac{B}{A} \quad \forall v \in M_K \setminus S. \quad (2)$$
Aèlic closure

Dehomogenize so that \( P = [\alpha, \beta, 1] \) and let

\[
S = M_{\infty}^\infty \cup \{ v \in M_K \mid |\alpha|_v \neq 1 \text{ or } |\beta|_v \neq 1 \}
\]

For \( v \notin S \), we have

\[
Q_v = \nu\text{-lim}_{n \in \mathbb{N}, P, Q} \varphi^n(P) = [x_v, y_v, 1] \in V(K_v)
\]

It follows that

\[
\nu\text{-lim}_{n \in \mathbb{N}, P, Q} \left( \frac{\alpha^k}{\beta^\ell} \right)^d = \frac{B}{A} \quad \forall v \in M_K \setminus S.
\]
The key proposition

Proposition

Let $\lambda, \xi \in K^*$. Assume that

- $\exists$ a finite $S \subset M_K$ including all archimedean places of $K$,
- $\exists$ infinite $\mathcal{N} \subset \mathbb{N}$,

such that

$$
\xi = \lim_{n \to \infty} \lambda^{d_n} \quad \forall v \in M_K \setminus S.
$$

Then, both $\lambda$ and $\xi$ are roots of unity.
Bang-Zsigmondy’s theorem

Theorem (Bang, Zsigmondy, Birkhoff-Vandiver, Postnikova-Schinzel)

Let $K$ be a number field, let $\lambda \in K^*$ be an element that is not a root of unity, and let

$$S_\lambda = M_\infty^\infty \cup \{v \in M_K : |\lambda|_v \neq 1\}.$$ 

For each $v \notin S_\lambda$, let $f_v(\lambda)$ denote the order of $\lambda$ in $\mathbb{F}_v^*$, the multiplicative group of residue field at $v$. Then the set

$$\mathbb{N} \setminus \{f_v(\lambda) : v \notin S_\lambda\}$$

is finite, i.e., all but finitely many positive integers occur as the order modulo $p$ of $\lambda$ for some prime $p$ of $K$. 
The proof of the key proposition

Recall Equation (3):

\[ \xi = v\lim_{n \to \infty} \lambda^{d^n} \quad \forall v \in M_K \setminus S. \]

Assume that \( \lambda \) is not a root of unity.
Then, Bang-Zsigmondy’s Theorem implies that

\[ \exists \text{ infinite sequence } n_1, n_2, n_3, \ldots \in \mathbb{N} \]

and for each \( i \), there exists \( v_i \in M_K \setminus S \) such that

\[ f_{v_i}(\lambda) = d^{n_i}. \]

We have

\[ \lambda^{d^{n_i}} \equiv \lambda^{f_{v_i}(\lambda)} \equiv 1 \pmod{p_i} \quad \text{for all } i = 1, 2, 3, \ldots. \]
The proof of the key proposition

Recall Equation (3):

\[ \xi = \lim_{n \to \infty} \lambda^{d_n} \quad \forall \nu \in M_K \setminus S. \]

Assume that \( \lambda \) is not a root of unity. Then, Bang-Zsigmondy’s Theorem implies that

\[ \exists \text{ infinite sequence } n_1, n_2, n_3, \ldots \in \mathcal{N} \]

and for each \( i \), there exists \( \nu_i \in M_K \setminus S \) such that

\[ f_{\nu_i}(\lambda) = d^{n_i}. \]

We have

\[ \lambda^{d^{n_i}} \equiv \lambda^{f_{\nu_i}(\lambda)} \equiv 1 \pmod{p_i} \quad \text{for all } i = 1, 2, 3, \ldots. \]
The proof of the key proposition

Recall Equation (3):

\[ \xi = \nu \text{-lim}_{n \to \infty} \lambda^{d_n} \quad \forall \nu \in M_K \setminus S. \]

Assume that \( \lambda \) is not a root of unity.

Then, Bang-Zsigmondy’s Theorem implies that

\[ \exists \text{ infinite sequence } n_1, n_2, n_3, \ldots \in \mathbb{N} \]

and for each \( i \), there exists \( \nu_i \in M_K \setminus S \) such that

\[ f_{\nu_i}(\lambda) = d^{n_i}. \]

We have

\[ \lambda^{d_{n_i}} \equiv \lambda^{f_{\nu_i}(\lambda)} \equiv 1 \pmod{p_i} \quad \text{for all } i = 1, 2, 3, \ldots. \]
The proof of the key proposition

Recall Equation (3):

\[ \xi = \nu\text{-lim}_{n \to \infty} \lambda^{d^n} \quad \forall v \in M_K \setminus S. \]

Assume that \( \lambda \) is not a root of unity.

Then, Bang-Zsigmondy’s Theorem implies that

\[ \exists \text{ infinite sequence } n_1, n_2, n_3, \ldots \in \mathcal{N} \]

and for each \( i \), there exists \( v_i \in M_K \setminus S \) such that

\[ f_{v_i}(\lambda) = d^{n_i}. \]

We have

\[ \lambda^{d^{n_i}} \equiv \lambda^{f_{v_i}(\lambda)} \equiv 1 \pmod{p_i} \quad \text{for all } i = 1, 2, 3, \ldots. \]
The proof of the key proposition

Recall Equation (3):

\[ \xi = \nu-\lim_{n \to \infty} \lambda^{d^n} \quad \forall \nu \in M_K \setminus S. \]

Assume that \( \lambda \) is not a root of unity. Then, Bang-Zsigmondy’s Theorem implies that

\[ \exists \text{ infinite sequence } n_1, n_2, n_3, \ldots \in \mathbb{N} \]

and for each \( i \), there exists \( \nu_i \in M_K \setminus S \) such that

\[ f_{\nu_i}(\lambda) = d^{n_i}. \]

We have

\[ \lambda^{d^{n_i}} \equiv \lambda^{f_{\nu_i}(\lambda)} \equiv 1 \pmod{p_i} \quad \text{for all } i = 1, 2, 3, \ldots. \]
For any given positive integer \( m \), we can find \( m \) distinct places \( v_1, \ldots, v_m \not\in S \) so that
\[
\lambda^{d^n} \equiv 1 \pmod{p_i} \quad \text{for all } 1 \leq i \leq m \text{ and all sufficiently large } n \in \mathbb{N}. 
\]

On the other hand, Equation (3) implies that
\[
\lambda^{d^n} \equiv \xi \pmod{p_i} \quad \text{for all sufficiently large } n \in \mathbb{N} \quad (4) 
\]
and all \( 1 \leq i \leq m \).

Hence,
\[
\xi \equiv 1 \pmod{p_i} \quad \text{for all } 1 \leq i \leq m. 
\]

As \( m \) is arbitrary, this forces \( \xi = 1 \).
For any given positive integer $m$, we can find $m$ distinct places $v_1, \ldots, v_m \not\in S$ so that

$$\lambda^d n \equiv 1 \pmod{p_i} \text{ for all } 1 \leq i \leq m \text{ and all sufficiently large } n \in \mathbb{N}.$$ 

On the other hand, Equation (3) implies that

$$\lambda^d n \equiv \xi \pmod{p_i} \text{ for all sufficiently large } n \in \mathbb{N} \quad (4)$$

and all $1 \leq i \leq m$.

Hence,

$$\xi \equiv 1 \pmod{p_i} \text{ for all } 1 \leq i \leq m.$$ 

As $m$ is arbitrary, this forces $\xi = 1$. 

Proof - continued

For any given positive integer \( m \), we can find \( m \) distinct places \( \nu_1, \ldots, \nu_m \not\in S \) so that

\[
\lambda^{d^n} \equiv 1 \pmod{p_i} \quad \text{for all } 1 \leq i \leq m \text{ and all sufficiently large } n \in \mathbb{N}.
\]

On the other hand, Equation (3) implies that

\[
\lambda^{d^n} \equiv \xi \pmod{p_i} \quad \text{for all sufficiently large } n \in \mathbb{N} \quad (4)
\]

and all \( 1 \leq i \leq m \).

Hence,

\[
\xi \equiv 1 \pmod{p_i} \quad \text{for all } 1 \leq i \leq m.
\]

As \( m \) is arbitrary, this forces \( \xi = 1 \).
Proof - continued

For any given positive integer \( m \), we can find \( m \) distinct places \( v_1, \ldots, v_m \not\in S \) so that
\[
\lambda^d n \equiv 1 \pmod{p_i} \quad \text{for all } 1 \leq i \leq m \text{ and all sufficiently large } n \in \mathbb{N}.
\]

On the other hand, Equation (3) implies that
\[
\lambda^d n \equiv \xi \pmod{p_i} \quad \text{for all sufficiently large } n \in \mathbb{N} \quad (4)
\]
and all \( 1 \leq i \leq m \).

Hence,
\[
\xi \equiv 1 \pmod{p_i} \quad \text{for all } 1 \leq i \leq m.
\]

As \( m \) is arbitrary, this forces \( \xi = 1 \).
For any given positive integer $m$, we can find $m$ distinct places $v_1, \ldots, v_m \not\in S$ so that
\[
\lambda^{dn} \equiv 1 \pmod{p_i} \quad \text{for all } 1 \leq i \leq m \text{ and all sufficiently large } n \in \mathbb{N}.
\]
On the other hand, Equation (3) implies that
\[
\lambda^{dn} \equiv \xi \pmod{p_i} \quad \text{for all sufficiently large } n \in \mathbb{N} \quad (4)
\]
and all $1 \leq i \leq m$.

Hence,
\[
\xi \equiv 1 \pmod{p_i} \quad \text{for all } 1 \leq i \leq m.
\]

As $m$ is arbitrary, this forces $\xi = 1$. 
Equation (3) becomes:

$$\nu \lim_{n \to \infty} \lambda^{d^n} = 1 \quad \text{for all } \nu \notin S.$$  \hspace{1cm} (5)

Apply Bang–Zsigmondy’s Theorem again to deduce that $\lambda$ is a root of unity.

- The main theorem (the case of power map) follows from applying the key proposition.
Equation (3) becomes:

\[
\nu\text{-lim}_{n \rightarrow \infty} \lambda^{d^n} = 1 \quad \text{for all } \nu \not\in S. \tag{5}
\]

Apply Bang–Zsigmondy’s Theorem again to deduce that \(\lambda\) is a root of unity.

- The main theorem (the case of power map) follows from applying the key proposition.
Abelian varieties and subvarieties

For the case of abelian subvariety, we apply an analogue of Bang-Zsigmondy’s theorem for elliptic curves.

**Theorem (Elliptic Zsigmondy Theorem)**

Let $E$ be an elliptic curve defined over $K$, and let $P \in E(K)$ be a nontorsion point, and let $S \subset M_K$ be a finite set of places including $M_K^\infty$ and all places of bad reduction for $E$. For each place $\nu \not\in S$, let $f_\nu(P)$ be the order of $P \pmod{p_\nu}$ in $E(\mathbb{F}_{p_\nu})$. Then, the set

$$\mathbb{N} \setminus \{f_\nu(P) : \nu \not\in S\}$$

is a finite set.

**Remark**

J. Silverman establishes the case for $K = \mathbb{Q}$. J. Cheon and S. Hahn prove the case for $K$ general number field.
Abelian varieties and subvarieties

For the case of abelian subvariety, we apply an analogue of Bang-Zsigmondy’s theorem for elliptic curves.

**Theorem (Elliptic Zsigmondy Theorem)**

Let $E$ be an elliptic curve defined over $K$, and let $P \in E(K)$ be a nontorsion point, and let $S \subset M_K$ be a finite set of places including $M_K^\infty$ and all places of bad reduction for $E$. For each place $v \notin S$, let $f_v(P)$ be the order of $P \pmod{p_v}$ in $E(\mathbb{F}_{p_v})$. Then, the set

$$\mathbb{N} \setminus \{f_v(P) : v \notin S\}$$

is a finite set.

**Remark**

J. Silverman establishes the case for $K = \mathbb{Q}$. J. Cheon and S. Hahn prove the case for $K$ general number field.
Proof of Theorem on translated subvarieties

Assume that there is a

\[ Q = (Q_v)_{v \in M_K} \in \overline{O_\varphi(P)} \cap V(A_K) \setminus O_\varphi(P) \cap V(K) \]

where \( V = B + T \) for some \( T \in A(K) \).

As in the case of power maps, we have an infinite set \( \mathcal{N} \) of positive integers so that

\[ Q_v = \nu \lim_{n \to \infty} [d^n]P. \]

Pass to the quotient \( E = A/B \) an elliptic curve over \( K \). We have

\[ \bar{Q}_v = \bar{T} \in E \]

where bar denotes images of elements in \( E = A/B \).
Proof of Theorem on translated subvarieties

Assume that there is a

$$Q = (Q_v)_{v \in M_K} \in \overline{\mathcal{O}_\varphi(P)} \cap V(A_K) \setminus \mathcal{O}_\varphi(P) \cap V(K)$$

where $V = B + T$ for some $T \in A(K)$.

As in the case of power maps, we have an infinite set $\mathcal{N}$ of positive integers so that

$$Q_v = v\text{-lim}_{\overset{n \in \mathcal{N}}{n \to \infty}} [d^n]P.$$ 

Pass to the quotient $E = A/B$ an elliptic curve over $K$. We have

$$\bar{Q}_v = \bar{T} \in E$$

where bar denotes images of elements in $E = A/B$. 
Our assumption says that

$$\nu-\lim_{n \to \infty} [d^n](\bar{P}) = \bar{T} \quad \forall \nu \in M_K.$$ 

The remaining argument is to apply the elliptic Zsigmondy’s theorem as in the case of power maps.
Question

1. Study the local-global question for the power map $\varphi$ and arbitrary hypersurface $V$ in $\mathbb{P}^N$.

2. For subvarieties of an abelian variety, can the assumption $V(K) = V(A_K) \cap A(K)$ be removed? i.e. $V(K)$ is Brauer-Manin unobstructed for arbitrary subvariety of $A$?

3. There is an obvious analogue for the same question in the setting of “$T$-modules”. We expect that our arguments can be applied to that case.
Question

1. Study the local-global question for the power map $\varphi$ and arbitrary hypersurface $V$ in $\mathbb{P}^N$.

2. For subvarieties of an abelian variety, can the assumption $V(K) = V(A_K) \cap A(K)$ be removed? i.e. $V(K)$ is Brauer-Manin unobstructed for arbitrary subvariety of $A$?

3. There is an obvious analogue for the same question in the setting of “$T$-modules”. We expect that our arguments can be applied to that case.
Question

1. Study the local-global question for the power map $\varphi$ and arbitrary hypersurface $V$ in $\mathbb{P}^N$.

2. For subvarieties of an abelian variety, can the assumption $V(K) = V(A_K) \cap \overline{A(K)}$ be removed? i.e. $V(K)$ is Brauer-Manin unobstructed for arbitrary subvariety of $A$?

3. There is an obvious analogue for the same question in the setting of “$T$-modules”. We expect that our arguments can be applied to that case.
Question

1. Study the local-global question for the power map $\varphi$ and arbitrary hypersurface $V$ in $\mathbb{P}^N$.

2. For subvarieties of an abelian variety, can the assumption $V(K) = V(A_K) \cap \overline{A(K)}$ be removed? i.e. $V(K)$ is Brauer-Manin unobstructed for arbitrary subvariety of $A$?

3. There is an obvious analogue for the same question in the setting of “$T$-modules”. We expect that our arguments can be applied to that case.
Thank you!