$K_{2i}O_F$ for $\mathbb{Z}_p$-extension

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Outline:

- Iwasawa’s Theorem
- Iwasawa’s Theorem for $K_{2n}O_F$
- Some Lemmas on $\Lambda$-modules
- The order of the $p$-primary part of $K_{2i}(O_{F_n})$
- $K$-groups and ideal class groups
Iwasawa’s Theorem

We recall a classical result from Iwasawa Theory.

Let $F$ be a number field. For a prime $p$, let $F_\infty/F$ be the cyclotomic $\mathbb{Z}_p$-extension and let $F_n$ be the unique intermediate field for $F_\infty/F$ such that $[F_n : F] = p^n$, $n \geq 0$. Let $p^{e_n}$ be the exact power of $p$ diving the class number of $F_n$. 
Iwasawa’s Theorem. There exist integers $\lambda \geq 0, \mu \geq 0$ and $\nu$, all independent of $n$, and an integer $n_0$ such that, for all $n \geq n_0$, 

$$e_n = \lambda n + \mu p^n + \nu.$$
Iwasawa’s Theorem for $K_{2n} O_F$

Let $F$ be a number field.

Assume that $\mu_p \subset F$ if $p > 2$ and $\mu_4 \subset F$ if $p = 2$.

Let $M$ be the maximal abelian $p$-extension of $F_\infty$ unramified outsider $p$. 
**Theorem.** For any \( i \geq 1 \), there exist integers \( n_i \) and \( \nu_i \) such that, for all \( n \geq n_i \),

\[
e(i)_n = \lambda n + \mu p^n + \nu_i,
\]

where \( p^{e(i)_n} = \#K_{2i}(O_{F_n})\{p\} \), \( \lambda \) and \( \mu \) are the classical Iwasawa invariants of the \( \Lambda \)-module \( \text{Gal}(M/F_\infty) \) independent of \( i \) and \( n \), and \( \nu_i \) is a constant independent of \( n \).

**Remark.** J. Coates, On \( K_2 \) and classical conjectures in algebraic number theory, Ann. Math., 95(1972), pp.99-116, proves the same assertion for \( i = 1 \).
Some Lemmas on $\Lambda$-modules

Let $F$ be a number field with degree $d$.

Assume that

$\mu_p \subset F$ if $p > 2$ and $\mu_4 \subset F$ if $p = 2$.

Let $q_0$ be the largest power of $p$ such that $\mu_{q_0} \subset F$.

Put $q_n = q_0 p^n$.

Write $F_n = F(\mu_{q_n})$ and $F_\infty = \bigcup_{n \geq 0} F_n$. 
Then $F_{\infty}/F$ is a $\mathbb{Z}_p$-extension, and as usual, we write $\Gamma = \text{Gal}(F_{\infty}/F)$, $\Gamma_n = \text{Gal}(F_{\infty}/F_n)$. Let

$$\kappa : \Gamma \longrightarrow 1 + q_0\mathbb{Z}_p$$

be the isomorphism determined by

$$\gamma(\zeta) = \zeta^{\kappa(\gamma)}, \quad \text{for all } \zeta \in W = \bigcup_{n \geq 0} \mu_{p^n}, \quad \gamma \in \Gamma.$$ 

Let $\Lambda = \mathbb{Z}_p[[T]]$ be the ring of formal power series in an indeterminate $T$ with coefficients in $\mathbb{Z}_p$. Choose, once and for all, a topological generator $\gamma_0$ of $\Gamma$. Then each compact $\Gamma$-module $X$ admits a unique structure of compact $\Lambda$-module such that

$$(1 + T)x = \gamma_0 x$$

for every $x$ in $X$. 
Let $\iota : \Lambda \to \Lambda$ be the automorphism given by

$$\iota\left( \sum_{m=0}^{\infty} c_m T^m \right) = \sum_{m=0}^{\infty} c_m \left( \kappa(\gamma_0)/(1 + T) - 1 \right)^m.$$

Given any $\Lambda$-module $Y$, denote by $Y^\bullet$ the $\Lambda$-module with the same underlying group as $Y$ but with $\Lambda$-module structure obtained from that of $Y$ by composition with $\iota$. 
Let $M$ be a $\Gamma$-module.


As $\mathbb{Z}_p$-module $M[n]$ is $M$; 

$\gamma$ action on $M[n]$ is given by the following:

For any $\gamma \in \Gamma$ and $x \in M$, $\gamma \ast x = \kappa(\gamma)^n \gamma(x)$. 

Thus $M[n]$ is isomorphic to $M(n)$ as $\Gamma$-modules.

For any $n \in \mathbb{Z}$, we put 

$$T^*_n = \kappa(\gamma_0)^n (1 + T) - 1.$$
Lemma 3.1. Let $\omega_n(T) = (1 + T)^{p^n} - 1$. For any non-zero element $g(T) \in \Lambda$, let $M$ denote the $\Lambda$-module $\Lambda/(g(T))$. And let $h : M \rightarrow M$ be the $\Lambda$-homomorphism given by multiplication by $\omega_n(T)$.

(1) (Lichtenbaum) $M[m]$ is isomorphic to $\Lambda/(g(T_{(m)}^*))$ as $\Lambda$-module.

(2) $h$ has a finite cokernel if and only if $\prod_{i=0}^n g(\zeta_{p^i} - 1) \neq 0$, and, if $\prod_{i=0}^n g(\zeta_{p^i} - 1) \neq 0$, the order of the cokernel is $\prod_{i=0}^n |g(\zeta_{p^i} - 1)|_{v_i}^{-1}$,
where the valuation $| \cdot |_{v_i}$ is the standard valuation of the field $\mathbb{Q}_p(\zeta_{p^i})$ such that $|\zeta_{p^i} - 1|_{v_i} = 1/p$ for all $i \geq 1$, and $| \cdot |_{v_0} = | \cdot |_p$ on $\mathbb{Q}_p$ such that $|p|_p = 1/p$.

(3) $h$ is injective if $\prod_{i=0}^{n} g(\zeta_{p^i} - 1) \neq 0$ or its kernel is infinite if $\prod_{i=0}^{n} g(\zeta_{p^i} - 1) = 0$. 
Lemma 3.2. For all \( h(T) \in \Lambda \) such that \( h(T) \) and \( \omega_n(T) \) are relatively prime, we have

\[
\# \frac{\Lambda}{(\omega_n(T), h(T))} = \prod_{i=0}^{n} |h(\zeta p^i - 1)|^{-1}_{v_i}.
\]
Let $M$ be a discrete $\Lambda$-module.

\[ \hat{M} = \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p) \] with $\Lambda$-action given by the following formula:

For $\lambda \in \Lambda$, $y \in M$, $\varphi \in \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$,

\[ (\lambda \varphi)(y) = \varphi(\lambda y). \]
Lemma 3.3. Let $M$ be a discrete $\Lambda$-module and assume that its Pontryagin dual $\widehat{M}$ is a finitely generated torsion $\Lambda$-module with no non-trivial finite $\Lambda$-submodule, and the following sequence is exact:

$$0 \rightarrow \widehat{M} \rightarrow \bigoplus_{j=1}^{r} \Lambda/(f_{j}(T)) \rightarrow D \rightarrow 0$$

where $D$ is a finite $\Lambda$-module. Put $f(T) = \prod_{j=1}^{r} f_{j}(T)$. Then the following assertions are equivalent for all integers $m$ and $n \geq 0$:

(i) $M(m)\Gamma_{n}$ is finite,  
(ii) $M(m)\Gamma_{n} = 0$,

(iii) $\prod_{i=0}^{n} f(\kappa(\gamma_{0})^{-m} \zeta_{p^{i}} - 1) \neq 0$.

If these assertions are valid, then the order of $M(m)\Gamma_{n}$ is

$$\prod_{i=0}^{n} |f(\kappa(\gamma_{0})^{-m} \zeta_{p^{i}} - 1)|_{v_{i}}^{-1}.$$
The order of the $p$-primary part of $K_{2i}(O_{F_n})$

$I_n$ (resp. $I$): the free abelian group generated by the primes of $F_n$ (resp. $F_\infty$) which do not lie above $p$.

$P_n$ (resp. $P$): the subgroup of principal $p$-ideals in $I_n$ (resp. $I$).

$C_n = I_n/P_n$ (resp. $C = I/P$).

$C_n$ (resp. $C$): the $p$-primary component of $C_n$ (resp. $C$).

$O_F$: the ring of integers in $F$.

$\mathcal{O}_0 = O_F[\frac{1}{p}]$ and $\mathcal{O}_n$ (resp. $\mathcal{O}$) is the algebraic closure of $\mathcal{O}_0$ in $F_n$ (resp. $F_\infty$).
\( \mathcal{U}_n \) (resp. \( \mathcal{U} \)): the group of all \( p \)-units in \( F_n \) (resp. \( F_\infty \)), i.e., the multiplicative group of the ring \( \mathcal{O}_n \) (resp. \( \mathcal{O} \)).

Then we have

\[ I = \lim_{\leftarrow} I_n, \quad \mathcal{C} = \lim_{\leftarrow} C_n, \quad \mathcal{C} = \lim_{\leftarrow} \mathcal{C}_n, \quad \mathcal{U} = \lim_{\leftarrow} \mathcal{U}_n. \]

There is a well defined surjective homomorphism

\[ \psi : (\mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}} F_\infty^{\times} \longrightarrow (\mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}} I. \]

We define \( \mathcal{M} \) to be its kernel.

Thus we have the exact sequence

\[ 0 \longrightarrow \mathcal{M} \longrightarrow (\mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}} F_\infty^{\times} \longrightarrow (\mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}} I \longrightarrow 0. \]
Lemma 4.1. (Soule) For any integer $i \geq 2$, one has

$$\mathcal{M}(i - 1)\Gamma_n = 0;$$

$$\mathcal{M}(i - 1)\Gamma_n = H^1(\mathcal{O}, W^{(i)})\Gamma_n$$

$$= H^1(\mathcal{O}_n, W^{(i)}) = (\mathbb{Q}_p / \mathbb{Z}_p)^{p^nd/2} \oplus G_{n,i},$$

where $G_{n,i}$ is a finite group.
Lemma 4.2. For any integers \( n \geq 0 \) and \( i \geq 1 \), we have

\[
K_{2i}(O_{F_n})\{p\} \cong G_{n,i+1}.
\]

This follows from

(1) \[
\mathcal{M}(i - 1)^{\Gamma_n} = (\mathbb{Q}_p/\mathbb{Z}_p)^{p^{n_d}/2} \oplus G_{n,i} \quad \text{(Soule)}.
\]

(2) Let \( O_S \) be the ring of \( S \)-integers in a number field \( F \) with some set \( S \) of finite places of \( F \). If \( p \) is a prime, then

\[
K_{2i}(O_S)\{p\} \cong H^2(O_S[\frac{1}{p}], \mathbb{Z}_p(i + 1))
\]
(Voevodsky, Rost, Suslin,..., See for example, C. Weibel’s paper in Handbook of K-Theory, editors: E.M.Friedlander and D.R.Grayson, Springer 2005.)

(3) For all integers $n \geq 0$ and $i \geq 2$,

$$H^2(\mathcal{O}_n, \mathbb{Z}_p(i)) \cong H^1(\mathcal{O}_n, W^{(i)})/H^1(\mathcal{O}_n, W^{(i)})_{\text{div}}.$$ 

(4) 

$$H^2(\mathcal{O}_n, \mathbb{Z}_p(i + 1)) \cong K_{2i}(\mathcal{O}_n)\{p\}$$ 

and  

$$K_{2i}(O_{F_n})\{p\} \cong K_{2i}(\mathcal{O}_n)\{p\}.$$
Let $f(T)$ be the characteristic polynomial of the $\Lambda$-module $\text{Gal}(M/F_\infty)^\bullet$.

**Theorem 4.3.** For any $n \geq 0$ and $i \geq 1$, we have

$$\#K_{2i}(O_{F_n})\{p\} = \#H(i)^{\Gamma^n} \cdot \prod_{j=0}^n |f(\kappa(\gamma_0)^{-i}\zeta_{p^j} - 1)|_{v_j}^{-1},$$

where

$$H = \frac{\Lambda^{d/2}}{\text{Gal}(M/F_\infty)^\bullet/t(\text{Gal}(M/F_\infty)^\bullet)}$$

is a finite $\Lambda$-module.
Corollary 4.4. If $S(F_∞/F) = 1$, i.e., $F_∞$ has only one prime divisor which is ramified for extension $F_∞/F$. Then for all integers $n \geq 0$ and $i \geq 1$, we have

$$\#K_{2i}(O_{F_n})\{p\} = \#H(i)^n \cdot \prod_{j=0}^{n} |h(\kappa(\gamma_0)^{-i} \zeta_{p^j} - 1)|_{v_j}^{-1},$$

where $h(T)$ is the characteristic polynomial of the Pontryagin dual of $\mathcal{C}$. 
Note that $H$ finite implies, for sufficiently large $n$, $H(i)\Gamma^n = H(i)$. So we have the following.

**Corollary 4.5.** Let $i \geq 1$. Then for sufficiently large $n$, we have

$$\#K_{2i}(O_{F_n})\{p\} = \#H \cdot \prod_{j=0}^{n} |f(\kappa(\gamma_0)^{-i} \zeta_{p^j} - 1)|_{v_j}^{-1}.$$ 

**Corollary 4.6.** The finite group $H$ is trivial if and only if there exists integer $i \geq 1$ such that

$$\#K_{2i}(O_F)\{p\} = |f(\kappa(\gamma_0)^{-i} - 1)|_p^{-1}.$$
Theorem 4.7. (1) For any $i \geq 1$, if $K_{2i}(O_F)\{p\} = 0$, then $K_{2i}(O_{F_n})\{p\} = 0$, for all $n \geq 0$.

(2) For any $i \geq 1$, there exist integers $n_i$ and $\nu_i$ such that, for all $n \geq n_i$,

$$e(i)_n = \lambda n + \mu p^n + \nu_i,$$

where $p^{e(i)_n} = \# K_{2i}(O_{F_n})\{p\}$, $\lambda$ and $\mu$ are the classical Iwasawa invariants of the $\Lambda$-module $\text{Gal}(M/F_\infty)$ independent of $i$ and $n$, and $\nu_i$ is a constant independent of $n$. 
**$K$-groups and ideal class groups**

In this section,

$p :$ an odd prime number;

$F = \mathbb{Q}(\zeta_p)$ the $p$-th cyclotomic field;

$F_n = \mathbb{Q}(\zeta_{p^{n+1}})$;

$F_\infty = \bigcup_{n \geq 0} F_n$;

$F^+ = \mathbb{Q}(\zeta_p)^+$;

$\Delta = \text{Gal}(F/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$;
\(\omega: \) the Teichmuller character;

\[ \Delta = \{ \omega^i | 0 \leq i \leq p - 2 \}; \]

\[ \varepsilon_i = \frac{1}{p-1} \sum_{a=1}^{p-1} \omega^i(a) \sigma_a^{-1}, \quad 0 \leq i \leq p - 2; \]

\[ \varepsilon_- = \frac{1-\sigma-1}{2} = \sum_{i \text{ odd}} \varepsilon_i; \]

\[ \varepsilon_+ = \frac{1+\sigma-1}{2} = \sum_{i \text{ even}} \varepsilon_i; \]

For an \(\mathbb{Z}_p[\Delta]\)-module \(A\),

\[ A(i) = \varepsilon_i A; \]

\[ A^- = \varepsilon_- A; \]

\[ A^+ = \varepsilon_+ A. \]
Recall that $M$ is the maximal abelian $p$-extension of $F_{\infty}$ unramified outsider $p$. Let $L$ denote the maximal unramified abelian $p$-extension over $F_{\infty}$ in $M$. Let $N'$ be the field generated over $F_{\infty}$ by the $p^{a}$-th roots of all elements $\varepsilon$ in $U$ for all integers $a \geq 0$.

K. Iwasawa, On $\mathbb{Z}_l$-extensions of algebraic number fields, Ann. Math. 98(1973), 246-326, shows the following:

(1) $\text{Gal}(M/N')^\bullet$ is isomorphic to the Pontryagin dual of $\mathcal{C}$ and it is a Noetherian torsion $\Lambda$-module with no non-trivial finite $\Lambda$-submodule.

(2) $\text{Gal}(N'/F_{\infty})^\bullet$ is isomorphic to the Pontryagin dual of $\mathcal{C}$, which is a torsion free $\mathbb{Z}_p$-module and is contained as a $\Lambda$-submodule of finite index in an elementary $\Lambda$-module of the form

$$\Lambda^{d/2} \oplus M$$
where $M = \bigoplus_{j=1}^{t} \Lambda/(g_j(T))$.

(3) Then the Galois group $\text{Gal}(M/F_\infty)$ is a Noetherian $\Lambda$-module and has no non-trivial finite $\Lambda$-submodule. We have

$$0 \rightarrow \text{Gal}(M/F_\infty)^\bullet/t(\text{Gal}(M/F_\infty)^\bullet) \rightarrow \Lambda^{d/2} \rightarrow H \rightarrow 0.$$
Now assume that $F = \mathbb{Q}(\zeta_p)$.

Let $K_n$ be the maximal unramified abelian $p$-extension over $F_n$ and $L_n$ be the maximal abelian extension over $F_n$ in $M$. Write

$$\omega_n = \omega_n(T) = (1 + T)^{p^n} - 1.$$

Then we have the following:

(i) $S(F_\infty/F) = 1$.

(ii) $C_n$ is also the ideal class group of $F_n$. And $C_n$ is also the $p$-primary subgroup of the ideal class group of $F_n$. 
(iii) 

\[ L_n = F_\infty K_n, \]

\[ \omega_n \text{Gal}(L/F_\infty) = \text{Gal}(L/L_n) \]

\[ (\text{Gal}(L/F_\infty))_{\Gamma_n} = \text{Gal}(L/F_\infty)/\omega_n \text{Gal}(L/F_\infty) \]

\[ \cong \text{Gal}(L_n/F_\infty) \cong \text{Gal}(K_n/F_n) \cong \mathfrak{c}_n. \]

(iv) 

\[ \text{Gal}(M/N')^\bullet \cong \text{Hom}(\mathfrak{c}, \mathbb{Q}_p/\mathbb{Z}_p) \cong \alpha(\text{Gal}(L/F_\infty)) \sim \text{Gal}(L/F_\infty), \]

where \( \alpha(\text{Gal}(L/F_\infty)) \) is the adjoint of \( \text{Gal}(L/F_\infty) \) and \( \sim \) means pseudo-isomorphism.
(v) Let $Y$ be the Pontryagin dual of $\mathcal{E}$. Then

$$Y \cong \text{Gal}(N'/F_\infty)^\bullet$$

and there is an exact sequence:

$$0 \rightarrow Y \rightarrow \Lambda \frac{p-1}{2} \rightarrow H \rightarrow 0,$$

where

$$H = \frac{\Lambda \frac{p-1}{2}}{\text{Gal}(M/F_\infty)^\bullet/t(\text{Gal}(M/F_\infty)^\bullet)}$$

is finite.

(vi) Let $f(T)$ be the characteristic polynomial of $\text{Gal}(L/F_\infty)$. Then $f(T)$ is also the characteristic polynomial of the $\Lambda$-module $\text{Gal}(M/N')^\bullet$ and $\text{Gal}(M/F_\infty)^\bullet$. 
(vii) Let $X = \text{Gal}(L/F\infty)$. Then $X^-$ has no non-trivial finite $\Lambda$-submodule and there are exact sequences:

$$0 \to A_i \to X^{(i)} \to \oplus_{j=1}^{t_i} \Lambda/(f_{i,j}(T)) \to B_i \to 0$$

where $A_i$ and $B_i$ are finite $\Lambda$-submodules and $A_i = 0$ if $i$ is odd.

Now set

$$A = \bigoplus_{i \text{ is even}} A_i$$

$$B^+ = \bigoplus_{i \text{ is even}} B_i, \quad B^- = \bigoplus_{i \text{ is odd}} B_i,$$

$$B = B^+ \oplus B^-,$$

$$f_i(T) = \prod_{j=1}^{t_i} f_{i,j}(T),$$
\[ f^+ = \prod_{i \text{ is even}} f_i(T), \]
\[ f^- = \prod_{i \text{ is odd}} f_i(T), \]
\[ \lambda = \lambda(X) = \deg f(T) \]
\[ \lambda_i = \lambda(X^{(i)}) = \deg f_i(T), \]
\[ \lambda^+ = \lambda(X^+) = \deg f^+(T), \]
\[ \lambda^- = \lambda(X^-) = \deg f^-(T), \]

Then
\[ f_i(T), \ f^+(T), \ f^-(T) \]
are the characteristic polynomials of the $\Lambda$-modules $X^{(i)}$, $X^+$ and $X^-$, respectively. So $$f(T) = \prod_{2 \leq i \leq p-2} f_i(T) = f^+(T)f^-(T)$$ and there are exact sequences:

$$0 \to A \to X^+ \to \bigoplus_{i \text{ is even}} t_i \bigoplus_{j=1} \Lambda/(f_{i,j}(T)) \to B^+ \to 0,$$

$$0 \to X^- \to \bigoplus_{i \text{ is odd}} t_i \bigoplus_{j=1} \Lambda/(f_{i,j}(T)) \to B^- \to 0,$$

$$0 \to A \to X \to \bigoplus_{2 \leq i \leq p-2} \bigoplus_{j=1} \Lambda/(f_{i,j}(T)) \to B \to 0.$$
(vi) If Vandiver’s conjecture holds for $p$, then
\[ X^{(i)} = \varepsilon_i X \cong \Lambda/(f(T, \omega^{1-i})) \]
for $i = 3, 5, \ldots, p-2$, where
\[ f((1+p)^s - 1, \omega^{1-i}) = L_p(s, \omega^{1-i}). \]
Factor $f(T, \omega^{1-i}) = p^{\mu_i} g_i(T) U_i(T)$ with $g_i$ distinguished if $g_i \neq 1$ and $U_i \in \Lambda^\times$. We know that $\mu_i = 0$. Therefore
\[ X^{(i)} \cong \Lambda/(g_i(T)) \]
and
\[ X = X^- \cong \bigoplus_{\{i \neq 1 \text{ odd}\}} \Lambda/(g_i(T)). \]

(viii) The finite $\Lambda$-module $H$ is trivial if and only if $H^1(\Gamma, \mathcal{U}) = 1$ if and only if Vandiver’s conjecture holds for $p$ which implies $A$ and $B$ are trivial and
Gal\((M/N')^\bullet\) \cong \text{Hom}(\mathcal{C}, \mathbb{Q}_p/\mathbb{Z}_p) \cong \alpha(X) \cong X \cong \bigoplus_{\{i \neq 1 \text{ odd}\}} \Lambda/(g_i(T)).

We can prove that the following exact sequence of \(\Lambda\)-modules:

\[
1 \longrightarrow G(M/N')^\bullet \longrightarrow G(M/F_\infty)^\bullet \longrightarrow G(N'/F_\infty)^\bullet \longrightarrow 1
\]

is split. So, we have

**Lemma 5.1.** The following sequence of \(\Lambda\)-modules is split:

\[
0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{M} \longrightarrow \mathcal{C} \longrightarrow 0,
\]

where \(\mathcal{E} = \mathcal{U} \otimes \mathbb{Q}_p/\mathbb{Z}_p\).
**Lemma 5.2.** Let $r \geq 1$ and $n \geq 0$. Then there are the following isomorphism

$$K_{2r}(O_{F_n})\{p\}^{(i)} \cong \mathcal{C}(r)^{\Gamma_n^{(i)}}, \quad i = 3, 5, \ldots, p - 2,$$

and exact sequence of abelian groups

$$0 \rightarrow H(r)^{\Gamma_n} \rightarrow K_{2r}(O_{F_n})\{p\}^+ \rightarrow \mathcal{C}(r)^{\Gamma_n^+} \rightarrow 0.$$
**Theorem 5.3.** (1) The odd prime number $p$ is regular if and only if there exist integers $i \geq 1$ and $n \geq 0$ such that $K_{2i}(O_{Fn})\{p\}$ is trivial, if and only if for all integers $i \geq 1$ and $n \geq 0$ such that $K_{2i}(O_{Fn})\{p\}$ is trivial.

(2) $\mathfrak{c}_0^{(i)} = 0$ if and only if

$\mathfrak{c}_n^{(i)} = 0$ for some $n \geq 0$ if and only if

$\mathfrak{c}_n^{(i)} = 0$ for all $n \geq 0$,

in the case, $\lambda_i = 0$.

Further more, if $i$ is odd, then $\lambda_i = 0$ implies $\mathfrak{c}_0^{(i)} = 0$. 
(3) Let $i = 3, 5, \cdots, p - 2$. Then

$K_{2r}(O_{F_0})\{p\}^{(i)} = 0$ if and only if

$K_{2r}(O_{F_n})\{p\}^{(i)} = 0$ for some $n \geq 0$ if and only if

$K_{2r}(O_{F_n})\{p\}^{(i)} = 0$ for all $n \geq 0$ if and only if

$\lambda_i = 0$.

$K_{2r}(O_{F_0})\{p\}^+ = 0$ if and only if

$K_{2r}(O_{F_n})\{p\}^+ = 0$ for some $n \geq 0$ if and only if

$K_{2r}(O_{F_n})\{p\}^+ = 0$ for all $n \geq 0$, and in this case, $\lambda^+ = 0$. 
Lemma 5.4. (a) Let \( M = \Lambda/(g(T)) \) with \( g(T) \) a distinguished polynomial and \( g(T) \) and \( \omega_n(T) \) relatively prime. For all integers \( i \geq 1 \) and \( n \geq 0 \), we have

\[
p-rk(M(i)_{\Gamma_n}) = p-rk(M_{\Gamma_n}) = \min\{p^n, \deg(g(T))\}.
\]

Moreover, let \( n_0 \) be the smallest integer such that \( p^{n_0} \geq \deg(g(T)) \). Then there exist integers \( n_1, n_2, \ldots, n_d \), where \( d = \deg(g(T)) \), such that for all \( n \geq n_0 + 1 \), we have

\[
\Lambda/(w_n, g) \cong \bigoplus_{i=1}^d p^{-n-n_i}Z_p/Z_p.
\]

(b) Let \( X \) be a Noetherian torsion \( \Lambda \)-module such that \( \mu(X) = 0 \) and \( X_{\Gamma_n} \) is finite for all \( n \geq 0 \). Then

\[
p-rk(X_{\Gamma_n}) \geq \lambda(X),
\]
\[ p\text{-rk}(X(i)\Gamma_n) \geq \lambda(X), \quad n \gg 0. \]

Furthermore, if \( X \) has no finite \( \Lambda \)-submodule, then for any integers \( i \), there exist integers \( n_0, n_1, n_2, \cdots, n\lambda(X), \nu_1, \nu_2, \cdots, \nu\lambda(X) \), such that for all \( n \geq n_0 \), we have

\[ p\text{-rk}(X\Gamma_n) = p\text{-rk}(X(i)\Gamma_n) = \lambda(X), \]

and

\[ X\Gamma_n \cong \bigoplus_{j=1}^{\lambda(X)} p^{-n-n_j} \mathbb{Z}_p / \mathbb{Z}_p, \]

\[ X(i)\Gamma_n \cong \bigoplus_{j=1}^{\lambda(X)} p^{-n-\nu_j} \mathbb{Z}_p / \mathbb{Z}_p. \]
Corollary 5.5. (1) Let \( i = 3, 5, \cdots, p - 2 \) be odd. Then there exist integers \( n_0, n_{i1}, \cdots, n_{i\lambda_i} \) such that for all \( n \geq n_0 \) we have

\[
p-rk(\mathcal{C}_n^{(i)}) = \lambda_i, \quad \mathcal{C}_n^{(i)} \cong \bigoplus_{j=1}^{\lambda_i} p^{-n-n_{ij}}\mathbb{Z}_p/\mathbb{Z}_p,
\]

hence

\[
p-rk(\mathcal{C}_n^-) = \lambda^-,
\]

\[
\mathcal{C}_n^- \cong \bigoplus_{3 \leq i \text{ is odd}} \bigoplus_{j=1}^{\lambda_i} p^{-n-n_{ij}}\mathbb{Z}_p/\mathbb{Z}_p, \quad n \gg 0.
\]

(2) Let \( i = 2, 4, \cdots, p - 3 \) be even. Then there exists integer \( n_0 \) such that for all \( n \geq n_0 \) we have

\[
p-rk(\mathcal{C}_n^{(i)}) \geq \lambda_i, \quad p-rk(\mathcal{C}_n^+) \geq \lambda^+.
\]
(3) Let $r \geq 1$ and $i = 3, 5, \cdots, p - 2$. There exist integers $n_0, n_1, \cdots, n_\lambda$ such that for all $n \geq n_0$, we have the following isomorphisms of abelian groups:

$$K_{2r}(O_{F_n})\{p\}^{(i)} \cong \bigoplus_{j=1}^{\lambda_i} p^{-n-n_j} \mathbb{Z}_p / \mathbb{Z}_p,$$

and

$$K_{2r}(O_{F_n})\{p\}^+ \cong H \bigoplus \bigoplus_{j=1}^{\lambda_+} p^{-n-m_j} \mathbb{Z}_p / \mathbb{Z}_p, \quad n \gg 0,$$

for some integers $m_1, \cdots, m_{\lambda+}$ independent of $n$.

(4)

$$p\text{-rk}(X_{\Gamma_n}) = p\text{-rk}(H) + p\text{-rk}(\alpha(X)_{\Gamma_n}),$$

hence

$$p\text{-rk}(c_{n^+}) = p\text{-rk}(H) + \lambda^+.$$
Corollary 5.6. Let \( i = 3, 5, \ldots, p - 2 \) be odd. Then \( \lambda_i = 1 \) if and only if \( C_n(i) \) is a cyclic group if and only if \( K_{2r}(O_{F_n})\{p\}(i) \) is a cyclic group, and in this case,

\[
C_n(i) \cong \frac{\mathbb{Z}_p}{(w_n(a_i))},
\]

\[
K_{2r}(O_{F_n})\{p\}(i) \cong \frac{\mathbb{Z}_p}{(w_n((1 + p)^{r(1 + a_i)} - 1))},
\]

where \( a_i \) is the root of the characteristic polynomial of \( X(i) \), i.e.,

\[
f(T, w^{1-i}) = (T - a_i)U_i(T), \quad U_i(T) \in \Lambda^*,
\]

where \( f((1 + p)^s - 1, \omega^{1-i}) = L_p(s, \omega^{1-i}) \).
Corollary 5.7. Let $\lambda$ denote the Iwasawa $\lambda$-invariant of the $\Lambda$-module $\text{Gal}(L/F_\infty)$. Then the following statements are equivalent:

(a) Vandiver conjecture holds for $p$;

(b) The finite $\Lambda$-module $H$ in is trivial;

(c) The finite $\Lambda$-modules $A$ and $B$ are trivial;

(d) $X$ is an elementary $\Lambda$-module;

(e) $\alpha(X) \cong X$;

(f) $p\text{-rk}(X_{\Gamma_n}) = p\text{-rk}(\alpha(X)_{\Gamma_n})$ for some $n \gg 0$;
(g) \( p\text{-rk}(X_{\Gamma_n}) = \lambda \) for some \( n \gg 0 \); 

(g') \( p\text{-rk}(X^+_{\Gamma_n}) = \lambda^+ \) for some \( n \gg 0 \); 

(h) \( p\text{-rk}(\text{Cl}(O_{F_n})) = \lambda \) for some \( n \gg 0 \); 

(h') \( p\text{-rk}(\text{Cl}(O_{F_n}^+)) = \lambda^+ \) for some \( n \gg 0 \); 

(j) for any \( i \geq 1 \), \( p\text{-rk}(K_{2i}(O_{F_n})) = \lambda \) for some \( n \gg 0 \); 

(j') for any \( i \geq 1 \), \( p\text{-rk}(K_{2i}(O_{F_n}^+)) = \lambda^+ \) for some \( n \gg 0 \). 

If these statements hold, then \( \lambda^+ = 0 \), i.e., \( \lambda = \lambda^- \).
Remarks. (1) (Kurihara) $\mathcal{C}_0^{(p-3)}$ always vanishes.

(2) (Soule) $\mathcal{C}_0^{(p-n)}$ is trivial if $\log p > n^{224n^4}$ odd.
**Theorem 5.8.** Let $p$ be an odd prime and assume Vandiver conjecture holds for $p$. Let $i_1, \ldots, i_s$ be the even indices $i$ such that $2 \leq i \leq p - 3$ and $p | B_i$. If

$$B_{1, \omega_{i-1}} \not\equiv 0 \mod p^2$$

and

$$\frac{B_i}{i} \not\equiv \frac{B_{i+p-1}}{i+p-1} \mod p^2 \text{ for all } i \in \{i_1, \ldots, i_s\},$$

then

(1)

$$X \cong \bigoplus_{i \in \{i_1, \ldots, i_s\}} \Lambda / (T - \alpha_i),$$

$$c_n \cong (\mathbb{Z}/p^{n+1}\mathbb{Z})^s, \text{ for all } n \geq 0,$$

where $\alpha_i \in p\mathbb{Z}_p$ and $v_p(\alpha_i) = 1$ for all $i \in \{i_1, \ldots, i_s\}$. 
(2) For all integers $m \geq 1$ and $n \geq 0$, we have

$$K_{2m}(O_{F_n})\{p\} \cong \bigoplus_{i \in \{i_1, \ldots, i_s\}} \mathbb{Z}/p^{n+1+c_i}$$

where $c_i = \nu_p((1 + p)^m(1 + \alpha_i) - 1) - 1$ for all $i \in \{i_1, \ldots, i_s\}$. In particular, if $m + \frac{\alpha_i}{p} \not\equiv 0 \mod p$ for all $i \in \{i_1, \ldots, i_s\}$, then

$$K_{2m}(O_{F_n})\{p\} \cong \text{Cl}(O_{F_n})\{p\} \cong (\mathbb{Z}/p^{n+1})^s.$$ 

Here $B_i$ and $B_{1,\omega^i-1}$ are respectively the ordinary Bernoulli numbers and the generalized Bernoulli numbers.
Corollary 5.9. Let $p$ be an odd prime and assume Vandiver conjecture holds for $p$. Let $i_1, \cdots, i_s$ be the even indices which satisfy conditions of Theorem 5.8. Then for all integers $n \geq 0$ and $m \geq 1$ such that $m \not\equiv -\frac{B_{1,\omega^i-1}}{B_{2,\omega^i-2/2-B_{1,\omega^i-1}}}$ (mod $p$) for all $i \in \{i_1, \cdots, i_s\}$, we have

$$K_{2m}(O_{F_n})\{p\} \cong \text{Cl}(O_{F_n})\{p\} \cong (\mathbb{Z}/p^{n+1}\mathbb{Z})^s.$$