Group Size and Bargaining Power

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We characterize a family of solutions for group bargaining problems axiomatically. Our axioms include four axioms that are equivalent to Nash's axioms for bargaining problems without groups: strong individual rationality, invariance with respect to affine transformation, anonymity, and independence of irrelevant alternatives. We use three additional axioms for the group bargaining problem: bilateral group consistency, replication invariance, and bilateral group size monotonicity. For this family of solutions, there exists some constant $\alpha \geq 0$ such that the "power" of a group is proportional to c^{α}, where c is the cardinality of the group. (*JEL* C71, C78).

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1 Introduction

In many, in fact most, applied bargaining problems, bargaining parties are not individuals but groups. The bargaining power of a group vis-*à*-vis other groups depends in general on the group's internal composition as well as those of the other groups.

Chae and Heidhues (2000) introduce a group bargaining problem, where groups of individuals bargain with each other, and propose a solution that extends the Nash (1950) solution. A bargaining problem *à la* Nash can be written as (N, S, d), where $N = \{1, \ldots, n\}$ is the set of individuals, $S \subset R^N$ is the feasible set[1], and $d \in R^N$ is a disagreement point, whereas a group bargaining problem can be written as (G, S, d), where $G = \{G_1, \ldots, G_m\}$ is a partition of N. The solution that Chae and Heidhues proposed can be characterized axiomatically by adding one axiom, called *Representation of Homogeneous Group*, to Nash's original four axioms. The axiom requires essentially that a homogeneous group can be replaced by a representative individual without changing the outcome. The solution maximizes the product of individual utilities raised to certain powers, where the power for an individual is the inverse of the cardinality of the group to which the individual belongs, *viz.*

$$\prod^{m}\left(\prod_{i \in G_j}(u_i - d_i)^{1/c_j}\right),$$

where $c_j = |G_j|$ (cardinality of group j). By contrast, the Nash solution that does not recognize the group partition maximizes

1. R^N denotes is the n-dimensional Euclidean space indexed by the set of individuals.

$$\prod_{i=1}^{n}(u_i - d_i).$$

The essential difference between the two solutions is that the solution that Chae and Heidhues propose treats each group equally while the Nash solution that ignores the group partition treats each individual equally. The crucial question is whether a group with more members should have greater power.

In this paper, we axiomatically characterize a family of solutions that includes both the solution that Chae and Heidhues propose and the Nash solution that ignores groups. Specifically, we show that any solution that satisfies *strong individual rationality (SIR), invariance with respect to affine transformation (IAT), anonymity (AN), independence of irrelevant alternatives (IIA), bilateral group consistency (BGC), replication invariance (RI),* and *bilateral group size monotonicity (BGSM),* to be defined precisely later, maximizes

$$\prod_{j=1}^{m}\left(\prod_{i\in G_j}(u_i - d_i)^{c_j^{\alpha-1}}\right)$$

for some $\alpha \geq 0$, where c_j is of the group's cardinality.

The first four axioms (SIR, IAT, AN, and IIA) are equivalent to Nash's original four axioms (Pareto efficiency, IAT, symmetry, and IIA) in the context of Nash's original bargaining problem without group partition. Since the group bargaining problem is more general than Nash's original bargaining problem, we need more axioms to narrow down solutions. The additional three axioms help to relate group sizes to group power.

For the class of solutions we characterize, c_j^{α} can be thought of as representing the power of group *j*. If $\alpha = 0$, as is the case with the solution Chae and Heidhues propose, the group size does not affect the group power. If $\alpha = 1$, as is the case with the Nash solution that ignores group partition, the group power is proportional to the group size. If $0 \leq \alpha < 1$, then the power of a group is less than proportional to its size, exhibiting the "joint bargaining paradox" of Harsanyi (1977).

Our methodology is a hybrid of the methodology of the axiomatic bargaining literature and the methodology of the surplus division literature. As in Nash (1950), we generalize the solution for a simplex to a general feasible set. In characterizing the solution for the simplex, we use a technique similar to that Moulin (1987) used to study a surplus division method that depends on the opportunity costs of agents. In our problem, groups' shares in a simplex problem depend on their cardinalities. Unlike opportunity costs, which can be any nonnegative numbers, cardinalities are integers. From the technical point of view, our innovation is to deal with this integer problem by identifying a positive integer with a finitely non-zero sequence of non-negative integers by prime factorization. This allows us to use Farka's lemma in the integer space.

2 The model and results

As in Nash (1950), we require that the feasible set S is compact, convex, and contains some payoff vector such that each individual's payoff is greater than the individual's breakdown payoff. A *solution* is a function F that associates to each bargaining problem (G, S, b) a feasible payoff

vector $F(G, S, b)$. Nash (1950)'s original four axioms are equivalent to the following four axioms in the context of the standard bargaining problems (N, S, b):

Strong Individual Rationality (SIR): $F_i(G, S, b) > b_i$ for all i.

Invariance with respect to Affine Transformation (IAT): *If L is an affine transformation on R^N (that is, there exist some numbers $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$, where $\beta_1, \ldots, \beta_n > 0$, such that $L(u_1, \ldots, u_n) = (\alpha_1 + \beta_1 u_1, \ldots, \alpha_n + \beta_n u_n)$ for any $u \in R^N$), then one has*

$$F(G, L(S), L(b)) = L(F(G, S, b)).$$

Anonymity (AN): $F(\phi(G), \phi(S), \phi(b)) = \phi(F(G, S, b))$ for any permutation $\phi: N \to N$.[2]

Independence of Irrelevant Alternatives (IIA): *If there exists another bargaining problem (G, S', b) such that S' is a subset of S and $F(G, S, b)$ belongs to S', then $F(G, S', b) = F(G, S, b)$.*

We will now introduce three additional axioms. For the first axiom, we reduce the feasible set fixing the payoffs of a subset of individuals. For any $K \subset N$, let $u_K \equiv (u_i)_{i \in K}$. The feasible set

2. Set $\phi(G) \equiv \{\phi(G_1), \ldots, \phi(G_m)\}$, where $\phi(G_j) = \{\phi(i) \in N; i \in G_j\}$ for $j = 1, \ldots, m$, and $\phi(S) = \left\{\phi(u) \in R^N; u \in S\right\}$, where $\phi(u) \equiv (u_{\phi(i)})_{i \in N}$ for any $u \in R^N$.

for K, given the payoffs $u_{N\setminus K}$ of individuals outside of K, is formally defined as the cross-section of the feasible set $S \subset R^N$ on the subspace R^K at $u_{N\setminus K}$, i.e.,

$$S_K(u_{N\setminus K}) = \{w_K \in R^K; \text{ there exists some } v \in S \text{ such that } v_{N\setminus K} = u_{N\setminus K} \text{ and } v_K = w_K\}.$$

Bilateral Group Consistency (BGC): *Let* $G = \{G_1, G_2, \ldots, G_m\}$ *and* $u = F(G, S, b)$.

Then $u_{G_1 \cup G_2} = F((\{G_1, G_2\}, S_{G_1 \cup G_2}(u_{N\setminus(G_1 \cup G_2)})), b_{G_1 \cup G_2})$.

Bilateral consistency means that if one reduces a bargaining problem by fixing the payoffs of individuals at the solution except for two groups, then the payoffs of the members of the two groups in the reduced bargaining problem are the same as at the solution of the original problem.

For the next axiom, we replicate of a group bargaining problem (G, S, b) by cloning each individual r times. Put

$$G_j^r \equiv \bigcup_{i \in G_j} \{i^1, \ldots, i^r\},\ N^r \equiv G_1^r \cup \ldots \cup G_m^r,\ G^r = \{G_1^r, \ldots, G_m^r\},$$

$$S^r \equiv \left\{ u \in R^{N^r};\ \text{there exists some } v \in S \text{ such that } v_{i^k} = v_i \text{ for } k = 1, \ldots, r, \text{ for } i \in N \right\},$$

and let b^r be a vector in R^{N^r} such that

$$b_{i^k}^r = b_i$$

for $k = 1, \ldots, r$, for $i \in N$.

Replication Invariance (RI): $F_{i^k}(G^r, S^r, b^r) = F_i(G, S, b)$ *for* $k = 1, \ldots, r$, *for* $i \in N$.

Replication invariance means that if we replicate the individuals and the feasible set r times, the payoffs of the individuals at the solution does not change. For the remaining axiom, we consider a simplex bargaining problem with two groups.

Put

$$\Gamma^l \equiv \left\{ x \in R^l;\ \sum_{i=1}^{l} x_i \le 1, x_i \ge 0 \text{ for } i = 1, \ldots, l \right\}$$

and

$$t\Gamma^l \equiv \{ tx \in R^l;\ x \in \Gamma^l \} = \left\{ y \in R^l;\ \sum_{i=1}^{l} y_i \le t, y_i \ge 0 \text{ for } i = 1, \ldots, l \right\} \text{ for } t > 0 .$$

Note that the Pareto frontier of Γ^l is the $(l-1)$-dimensional unit simplex Δ^{l-1}.

Bilateral Group Size Monotonicity (BGSM): Let $G = \{G_1, G_2\}$ and $c_1 \ge c_2$ then

$$\sum_{i \in G_1} F_i(G, \Gamma^n, 0) \ge \sum_{i \in G_2} F_i(G, \Gamma^n, 0) .$$

Bilateral group size monotonicity means that in a symmetric cake-splitting problem with two groups, the share of the larger group is at least as great as that of the smaller group.

Theorem 1. *A solution F satisfies SIR, IAT, IIA, AN, BGC, RI, and GSM if and only if* $F(G, S, b)$ *solves the maximization problem*

$$Max_{u \in S,\, u \geq b} \prod_{j=1}^{m} \left(\prod_{i \in G_j} (u_i - b_i)^{c_j^{\alpha - 1}} \right) \tag{1}$$

for some $\alpha \geq 0$.

Proof. Since S is compact and convex, there exists a unique solution to the maximization problem (1). Denote it by $F^*(G, S, b)$. It is easy to check that F^* satisfies SIR, IAT, IIA, AN, BGC, RI, and GSM. We will show that if F satisfies these properties, then $F = F^*$.

Consider the group bargaining problem $(G, \Gamma^n, 0)$. Since SIR and IAT imply PE, $F(G, \Gamma^n, 0)$ belongs to the interior of the $(n-1)$-dimensional simplex Δ^{n-1}. AN implies that $F(G, \Gamma^n, 0)$ depends only on the configuration of group sizes $(c_1, \ldots, c_m)$ and that for each $j = 1, \ldots, m$, there exists some v_j such that

$$F_i(G, \Gamma^n, 0) = v_j \text{ for any } i \in G_j.$$

Let P be the set of positive integers and set $P^m \equiv P \times \ldots \times P$. Define a function, for $m = 2, 3, \ldots,$

$$\varphi^m = (\varphi_1^m(c_1, \ldots, c_m), \ldots, \varphi_m^m(c_1, \ldots, c_m))$$

from P^m to the interior of the $(m-1)$-dimensional simplex Δ^{m-1} by

$$\varphi_j^m(c_1, \ldots, c_m) \equiv \sum_{i \in G_j} F_i(G, \Gamma^n, 0) = c_j v_j \text{ for } j = 1, \ldots, m.$$

We will show below that there exists some constant $\alpha \geq 0$ such that

$$\varphi^m(c_1, \ldots, c_m) = \left(\frac{c_1^\alpha}{c_1^\alpha + \ldots + c_m^\alpha}, \ldots, \frac{c_m^\alpha}{c_1^\alpha + \ldots + c_m^\alpha} \right). \tag{2}$$

so that

$$F_i(G, \Gamma^n, 0) = \frac{c_j^{\alpha-1}}{c_1^\alpha + \ldots + c_m^\alpha} \text{ for any } i \in G_j.$$

For a general bargaining problem (G, S, b), by IAT, there exists a unique affine transformation of utilities, L, so that $L(b) = 0$ and $L(F^*(G, S, b)) = F(G, \Gamma^n, 0)$. Then $L(S)$ is supported by the unit simplex Δ^{n-1} at $F(G, \Gamma^n, 0)$. Thus IIA implies that $F(G, L(S), 0) = F(G, \Gamma^n, 0)$. By IAT, $F(G, S, b) = F^*(G, S, b)$. Thus we now only have to derive (2).

Let $F(G, \Gamma^n, 0) = u$. Then by BGC and IAT,

$$\begin{aligned} u_{G_1 \cup G_2} &= F((\{G_1, G_2\}, \Gamma^n{}_{G_1 \cup G_2}(u_{N\setminus(G_1 \cup G_2)})), 0) \\ &= F(\{G_1, G_2\}, (\Sigma_{i \in G_1 \cup G_2} u_i)\Gamma^2, 0) \\ &= (\Sigma_{i \in G_1 \cup G_2} u_i)(F(\{G_1, G_2\}, \Gamma^2, 0)). \end{aligned}$$

Thus

$$\frac{\varphi_1^m(c_1, \ldots, c_m)}{\varphi_2^m(c_1, \ldots, c_m)} = \frac{\varphi_1^2(c_1, c_2)}{\varphi_2^2(c_1, c_2)} \equiv h(c_1, c_2) \ .$$

By AN, one has

$$\frac{\varphi_2^m(c_1, \ldots, c_m)}{\varphi_3^m(c_1, \ldots, c_m)} = h(c_2, c_3) \quad \text{for } m \geq 3 .$$

Thus $h(c_1, c_2) \cdot h(c_2, c_3) \cdot h(c_3, c_1) = 1$, which implies

$$h(c_1, c_2) = \frac{h(c_1, c_3)}{h(c_2, c_3)} .$$

Fix $c_3 = 1$ (or fix $c_3 = l$ at any arbitrary $l \in P$) and define $g(c_i) \equiv h(c_i, 1)$. Then

$$h(c_1, c_2) = \frac{g(c_1)}{g(c_2)}$$

and thus

$$\frac{\varphi_i^m(c_1, \ldots, c_m)}{\varphi_j^m(c_1, \ldots, c_m)} = \frac{g(c_i)}{g(c_j)} \text{ for any } i, j \leq m .$$

Since $\varphi^m(c_1, \ldots, c_m) \in \Delta^{m-1}$, one has

$$\varphi_i^m(c_1, \ldots, c_m) = \frac{g(c_i)}{g(c_1) + \ldots + g(c_m)}.$$

By RI, AN, and IAT,

$$\begin{aligned} F_i(G, \Gamma^n, 0) &= F_{i^k}(G^r, (\Gamma^n)^r, 0) \\ &= F_{i^k}(G^r, r\Gamma^{rn}, 0) \\ &= rF_{i^k}(G^r, \Gamma^{rn}, 0) \end{aligned}$$

for $k = 1, \ldots, r$, for $i \in N$. Thus

$$\frac{g(c_j)}{g(c_1) + \ldots + g(c_m)} = \frac{g(rc_j)}{g(rc_1) + \ldots + g(rc_m)} \text{ for any } r \in P, j = 1, \ldots, m.$$

and thus

$$\frac{g(rc_1)}{g(c_1)} = \frac{g(rc_2)}{g(c_2)} \text{ for any } r, c_1, c_2 \in P.$$

Without loss of generality, we can set $g(1) = 1$. So we have

$$\frac{g(xy)}{g(y)} = \frac{g(x)}{g(1)} = g(x) \text{ for any } x, y \in P,$$

i.e.,

$$g(xy) = g(x)g(y) \text{ for any } x, y \in P,$$

and thus

$$g(xx') = g(x)g(x') \text{ for any positive integers } x, x'.$$

Let p_i ($i = 1, 2, \dots$) be the i-th prime number, i.e., $p_1 = 2, p_2 = 3, p_3 = 5, \dots$. Then any positive integer x can be written as

$$x = \prod_{i=1}^{\infty} p_i^{x_i} = e^{\sum_{i=1}^{\infty} x_i \log p_i}$$

for some nonnegative integers x_i ($i = 1, 2, \dots$).

Let Z be the set of finitely non-zero sequences of integers $z = (z_1, z_2, \dots)$ and define a real-valued function Λ on Z by

$$\Lambda(z) \equiv \log g\left(e^{\sum_{i=1}^{\infty} z_i \log p_i}\right).$$

Then one has

$$\Lambda(z) = \sum_{i=1}^{\infty} \lambda_i z_i,$$

where $\lambda_i = \log g(p_i)$ for $i = 1, 2, \dots$.

By BGSM, $x \geq y$ implies $g(x) \geq g(y)$ and thus $\Lambda(x) \geq \Lambda(y)$. Rewriting,

$$\sum_{i=1}^{\infty} (\log p_i)(x_i - y_i) \geq 0 \text{ implies } \sum_{i=1}^{\infty} \lambda_i (x_i - y_i) \geq 0,$$

i.e.,

$$\sum_{i=1}^{\infty} (\log p_i) z_i \geq 0 \text{ implies } \sum_{i=1}^{\infty} \lambda_i z_i \geq 0 \text{ for any } z \in Z.$$

Thus, by Farkas's Lemma, there exists some $\alpha \geq 0$ such that

$$(\lambda_1, \lambda_2, \ldots) = \alpha(\log p_1, \log p_2, \ldots)$$

and thus $g(x) = e^{\Sigma_{i=1}^{\infty} \lambda_i z_i} = e^{\alpha \Sigma_{i=1}^{\infty} (\log p_i) x_i} = x^{\alpha}$ for any positive integer x. ***Q.E.D.***

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