# ANALYSIS OF THE SEVERAL NUMERICAL SCHEMES FOR THE THIN FILM EQUATIONS

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**ABSTRACT**

Equations of the type $h_t + (h^2 - h^3)_x = -\epsilon^3 (h^3 h_{xxx})_x$ arise in the context of thin films driven the competing effects of a induced surface tension gradient and gravity. These films arise in thin coating flows and are of great technical and scientific interest. Recently, it has been discovered that the interfacial dynamics of these films includes the development of undercompressive shocks. There is also a growing amount of theoretical work indicating that undercompressive shocks are observed in other physical systems. Here we focus on the several numerical methods to apply the model equation and the comparison and analysis of the numerical results. Since we do not know the exact solution of the model equation we choose one of the schemes and have a approximate solution using very finer grids. In order to accurately compare the solutions we are using an one dimensional model problem.

## INTRODUCTION

A phenomenon of interest in liquid thin film flows under certain conditions is driven by competing effects of the gravity and a thermally induced surface tension gradient. Understanding the thin liquid film is important in many physical systems such as coating processes, de-icing of airplane wings and construction of photographic film. Recently, it has been discovered that the interfacial dynamics of this liquid thin film includes the development of undercompressive shocks. Detailed numerical simulation explain this theory. Several finite difference schemes have been developed for the simulation of the liquid thin film flows and are available in the literature. The main difference among them is the way they address the problem of shock waves formation. It is not easy to treat the fourth order diffusion terms, which is of fundamental importance in the accuracy and effectiveness of the calculations. The results from such simulations can aid in design of, for instance, new materials and device structures.

The model equation is constructed by the fourth order regularization and the nonconvex flux function. Numerical schemes for the solution of nonconvex flux has been successful thanks to seminal numerical schemes such as high order Godunov, WENO-LF and central-upwind schemes. However, it is not proper to treat the the fourth order diffusion terms with those schemes directly using explicit methods because of several limitations. Solving the diffusion term using explicit methods it take too much cost to compute it. Because of the fourth order differential equations it is needed very smaller time steps which is in the ratio of $(dt)^4$. In order to treat the diffusion terms we have to concern implicit methods.

We consider to solve the model equation using numerical schemes in two things. One is using implicit methods for the flux and the fourth order diffusion term both. Another is using explicit methods for the flux and implicit methods for the diffusion with splitting methods. After we are obtained the linear system using implicit methods we solve it by direct methods (e.g. gaussian elimination) or iterative methods.

In this paper, we compare approximate solutions obtained by numerical solutions and discuss effectiveness and accuracy of the numerical schemes. Since we do not know the exact solution of the model equation we choose one of the schemes and have a approximate solution using very finer grids. In order to accurately compare the solution we are using an one dimensional model problem. It is hard to compare a two dimensional problem exactly without knowing the exact solution of the model equation.

## MODEL EQUATION

We model the dynamics of a thin liquid film on an inclined substrate, driven by the combined effects of a thermally induced Marangoni force and gravity. After a suitable rescaling and calculating the thin film motion by a partial differential equation(PDE) for thickness $h(x,t)$ of the film above the inclined plane, as a function of distance $x$ down the plane, and the time $t$ is the following [2] :

$$h_t + (h^2 - h^3)_t = -\epsilon^3 (h^3 h_{xxx})_x, \tag{1}$$

in which $\epsilon > 0$ is a small parameter.

## NUMERICAL METHODS AND RESULTS

In numerical computations, a convection-diffusion equation such as (1) is usually treated by a fractional step splitting method, in which one alternates between solving the convection equation

$$h_t + (h^2 - h^3)_t = 0, \tag{2}$$

and the diffusion equation

$$h_t = -\epsilon^3 (h^3 h_{xxx})_x, \tag{3}$$

in each time step. The flux function $f(h) = h^2 - h^3$ is the nonconvex and has a global maximum at $h_{\max} = \frac{2}{3}$ and an inflection point at $h_I = \frac{1}{3}$.

The numerical method that we use is based on finite difference discretization method with $N$ uniformly spaced grid points. In order to solve the equation (1) we consider two ways. One is using implicit methods directly without splitting. Well known fully implicit and Crank-Nicolson methods are used with proper initial and boundary conditions. Then we obtain the linear system $Ax = b$ where $A$ is *upper bandwidth 3* and *lower bandwidth 3*. We are using direct methods or iterative methods for the linear system and compare the effectiveness of them.

Another method is using explicit methods for convection (2) and fully implicit or Crank-Nicolson methods for the fourth order diffusion (3). Here we are using higher order Godunov method with proper limiter function, WENO-LF and central-upwind schemes for equation (2).

Briefly we describe the basic ideas of the explicit schemes which involve the computation of the cell-interface values of an one-dimensional function from its cell averages at the neighboring cells. The Godunov scheme [1,3] is based on either the exact or an approximate solution of the Riemann problem using characteristic information within the framework of a conservative method. The Godunov method can be modified to a second-order scheme by employing a proper limiter (e.g. Super bee, Van Leer limiter). The central-upwind scheme [4,3] is constructed based on a piecewise linear approximation with proper limiter functions. Since the speed of propagation is related to the CFL condition, we can estimate the local speeds of the right and left side at the cell boundary. Using the local speeds at the interface $x = x_{j+1/2}$ we construct the numerical flux function. Finally we describe the $5^{th}$ order WENO (weighted essentially non-oscillatory)-LF[5,3]. First we choose five-point stencil and make 3 polynomial functions $h^0, h^1$, and $h^2$ using three sub-stencils of the given stencil. After that we construct the new polynomial function from $h^0, h^1$, and $h^2$ with weight properly. To avoid entropy violating solutions and obtain the numerical stability we split the flux $f(h)(= h^2 - h^3)$ into two components $f^+$ and $f^-$ such that $f(h) = f^+(h)+f^-(h)$ where $\partial f^+/\partial h \geq 0$ and $\partial f^-/\partial h \leq 0$. One of the simplest flux splitting is the Lax-Friedrichs splitting which is given by $f^{\pm} = (f(h) \pm \alpha h)/2$, where $\alpha = \max|f'(h)|$ over the pertinent range of $h$ which can be decided a priori using the explicit formula for the exact solution.

Table 1 The Maximum values and the difference of exact(max(h))- approximate(max(h)))

| Method \ grid | | 100 | 200 | 400 | 800 |
|---|---|---|---|---|---|
| WENO-LF | Max | 0.3707 | 0.3973 | 0.4062 | 0.4071 |
| | Diff | 0.0365 | 0.0099 | 0.0010 | 0.0001 |
| Central-upwind | Max | 0.3746 | 0.4011 | 0.4068 | 0.4072 |
| | Diff | 0.0326 | 0.0061 | 0.0004 | 0. |
| High order Godunov | Max | 0.3927 | 0.4054 | 0.4074 | 0.4073 |
| | Diff | 0.0145 | 0.0018 | -0.0002 | -0.0001 |
| Fully-Implicit | Max | 0.4004 | 0.4040 | 0.4065 | 0.4071 |
| | Diff | 0.0068 | 0.0032 | 0.0007 | 0.0001 |
| Crank-Nicolson | Max | 0.4117 | 0.4103 | 0.4102 | 0.4089 |
| | Diff | -0.0045 | -0.0031 | -0.0030 | -0.0017 |

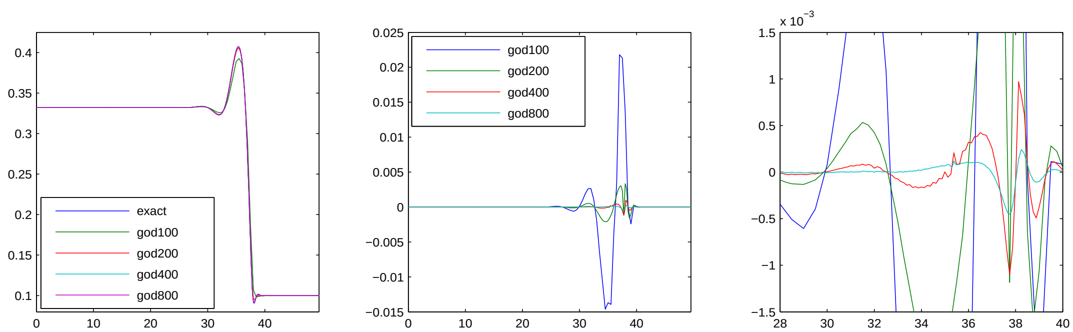


Figure 1. Compare of high order Godunov-iterative methods with grids 100,200,400 and 800 and WENO-LF-iterative with 2400 grids

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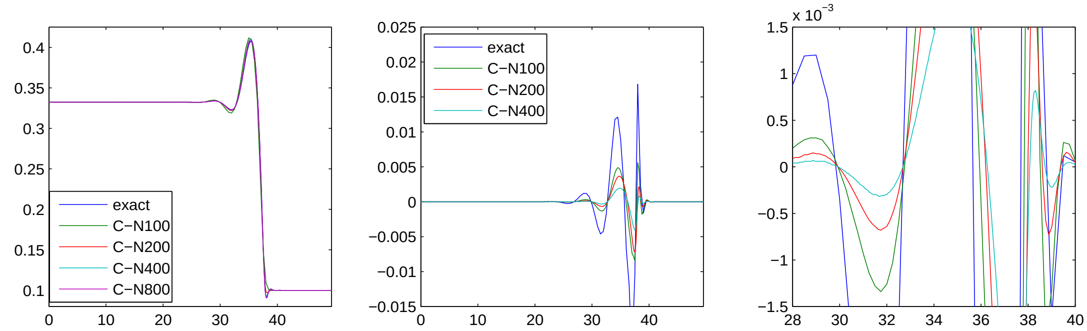


Figure 2. Compare of high order Crank-Nicolson-iterative methods with grids 100,200,400 and 800 and WENO-LF-iterative with 2400 grids

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