ERROR ANALYSIS OF ONE-DIMENSIONAL HELMHOLTZ EQUATION WITH PML BOUNDARY

Taeyoung HA [1] and Imbunm KIM [1]

1) *Department of Mathematical Sciences, Seoul National University, Seoul 151-747, KOREA*

Corresponding Author : Taeyoung Ha, tyha@math.snu.ac.kr

ABSTRACT

In this paper, the linear conforming finite element method for the one-dimensional Bérenger's PML boundary is investigated and well-posedness of the given equation is discussed. Furthermore, optimal error estimates and stability in the the L^2 or H^1-norm are derived under the assumption that h, $h^2\omega^2$ and $h^2\omega^3$ are sufficiently small, where h is the mesh size and ω denotes a fixed frequency. Numerical examples are presented to validate the theoretical error bounds.

INTRODUCTION

The idea of a perfectly matched layer(PML) was introduced by Bérenger [1]. It is intended for constructing the absorbing layer in the truncated computational domain. The PML technique is now widely used for simulating the propagation of waves in unbounded domains, particularly in the field of acoustics, elastodynamics and electromagnetics[2,3,5–7]. They studied the behavior of exact solution of the wave equation, elastic equation and Maxwell equation with PML in the time or frequency domain and the stability and error estimate with respect to the parameters of the layers.

However, to our knowledge, there has not been any error analysis, nor stability done on the wave equation with Bérenger PML boundary in frequency domain. Therefore, in this paper, we will attempt to demonstrate the existence and uniqueness of the solution for the wave equation with Bérenger PML in the frequency domain and find the regularity order in error analysis and stability in H^1-norm. The difficulties in finding the regularity coefficient is due to the fact that the bilinear form associated with the problem is not coercive. To overcome this difficulty, we used the boot-strapping argument [4,8–10] and attempted to obtain the regularity estimate depending on the frequency and damping function.

PML HELMHOLTZ EQUATION AND WEAK FORMULATION

One dimensional Helmholtz problem can be obtained in the frequency domain as

$$-\omega^2\hat{u}(x,\omega) - \frac{\partial^2\hat{u}(x,\omega)}{\partial x^2} = \hat{f}(x,\omega), \quad x \in \mathcal{R}^1, \tag{1}$$

$$\lim_{r\to\infty} r\left(\frac{\partial\hat{u}(x,\omega)}{\partial r} - i\omega\hat{u}(x,\omega)\right) = 0, \tag{2}$$

where $r = |x|$ and

$$\hat{u}(x,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega t}dt.$$

For the sake of notation brevity, we write $\hat{u}(x,\omega)$ as $u(x,\omega)$.By setting $\Omega_c = (0,1)$ and $\Omega_\infty = (-\epsilon, 1+\epsilon)$ with $\mathrm{e} > 0$ being the open ball containing Ω_c, we let

$$\tilde{x}(x,\omega) = x + \frac{i}{\omega}\int_0^x \xi(s)ds, \quad x \in \Omega_\infty, \tag{3}$$

where $\xi(x) = 0$ in Ω_c and $\xi(x)$ are the smooth, nonzero and nonnegative function. From this,it is easy to confirm that

$$\frac{\partial \tilde{x}(x,\omega)}{\partial x} = 1 + \frac{i}{\omega}\xi(x) := \tilde{\xi}(x,\omega). \tag{4}$$

The truncated PML Helmholtz equation is as follows :

$$-\omega^2 u(x,\omega)\tilde{\xi}(x,\omega) - \frac{\partial}{\partial x}\left(\frac{1}{\tilde{\xi}(x,\omega)}\frac{\partial u(x,\omega)}{\partial x}\right) = f(x,\omega), \quad x \in \Omega_\infty, \tag{5}$$
$$u(x,\omega) = 0, \quad x \in \partial\Omega_\infty.$$

Set the sesquilinear form $\Lambda_\omega(\cdot,\cdot)$ as

$$\Lambda_\omega(v,w) = -\omega^2\int_\Omega v(x)\overline{w(x)}\tilde{\xi}(x,\omega)dx + \int_\Omega \frac{\partial v(x)}{\partial x}\frac{\overline{\partial w(x)}}{\partial x}\frac{1}{\tilde{\xi}(x,\omega)}dx.$$

Then, a weak form of (5) can be defined by finding a solution $u(\cdot,\omega) \in H_0^1(\Omega_\infty)$ of

$$\Lambda_\omega(u(\cdot,\omega),v) = (f(\cdot,\omega),v), \quad v \in H_0^1(\Omega_\infty). \tag{6}$$

ERROR ESTIMATES FOR CONFORMING FINITE ELEMENT METHOD

Dividing Ω_∞ into the subdivisions $[x_j, x_{j+1}]$, $j = 0,\cdots,N-1$ with $x_0 = -\,\mathrm{e}$ and $x_N = 1+\mathrm{e}$, and $V_h = \{v \in C^0(\Omega_\infty) \mid v \in P_1([x_j,x_{j+1}]),\ j = 0,\cdots,N-1,\ v(x_0) = v(x_N) = 0\}$, where P_1 is the space of polynomials of degree 1 or less on Ω_∞ and $h = (1+2\,\mathrm{e})/N$. The discretized formulation of approximation solution can be written as follows : find $u_h(\cdot,\omega) \in V_h$ such that

$$\Lambda_\omega(u_h(\cdot,\omega),v) = (f(\cdot,\omega),\ v), \qquad v \in V_h.$$

Theorem 0.1. *Suppose $\omega > 0$ and $h,\ h^2\omega^2,\ h^2\omega^3$ are small. Then*
(a) $||(u-u_h)(\cdot,\omega)||_0 \le C||\tilde{\xi}(\cdot,\omega)||_\infty^2\left(||\tilde{\xi}(\cdot,\omega)||_\infty + \omega||\tilde{\xi}(\cdot,\omega)||_\infty^2\right)^2||f(\cdot,\omega)||_0h^2$,
(b) $\left|\left|\frac{\partial(u-u_h)}{\partial x}(\cdot,\omega)\right|\right|_0 \le C||\tilde{\xi}(\cdot,\omega)||_\infty^2\left(||\tilde{\xi}(\cdot,\omega)||_\infty + \omega||\tilde{\xi}(\cdot,\omega)||_\infty^2\right)||f(\cdot,\omega)||_0h,$
where C_1 and C_2 are dependent on only ξ.

Remark 0.2. *If ω is sufficiently large, then $||\tilde{\xi}(\cdot,\omega)||_\infty \approx 1$. Therefore, for large ω, Theorem 0.1 becomes*

$$(a)||(u-u_h)(\cdot,\omega)||_0 \le C\,(1+\omega)^2\,||f(\cdot,\omega)||_0h^2$$
$$(b)\left|\left|\frac{\partial(u-u_h)}{\partial x}(\cdot,\omega)\right|\right|_0 \le C\,(1+\omega)\,||f(\cdot,\omega)||_0h.$$

This is the same result of Douglas et al[4].

REFERENCES

1. J. P. Berenger, *A perfectly matched layer for the absorption of electromagnetic waves*, J. Comput. Phys. **114** (1994), 184–200.
2. G. C. Cohen, *Higher-order numerical methods for transient wave equations*, Scientific Computation, Springer-Verlag, Berlin, 2002.
3. J. Diaz and P. Joly, *A time domain analysis of PML models in acoustics*, Comput. Methods Appl. Mech. Engrg. *to appear* (2005).
4. J. Douglas, Jr., J. E. Santos, D. Sheen and Lynn S. Bennethum, *Frequency Domain Treatment of One–Dimensional Scalar Waves*, Math. Models Meth. Appl. Sci. **3(2)** (1993), 171–194.
5. F. Ihlenburg, Finite element analysis of acoustic scattering, Applied Mathematical Sciences, **132**, Springer-Verlag, New York, 1998.
6. P. Monk, *Finite element methods for Maxwell's equations*, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2003.
7. B. Sjågreen and N. A. Petersson, *Perfectly matched layers for Maxwell's equations in second order formulation*, J. Comput. Phys. **209** (2005), 19–46.
8. J. Douglas, Jr., J. E. Santos and D. Sheen, *Approximation of Scalar Waves in the Space–Frequency Domain*, Math. Models Meth. Appl. Sci. **4(4)** (1994), 509–531.
9. J. Douglas, Jr., J. E. Santos, D. Sheen, and X. Ye, *Nonconforming Galerkin methods based on quadrilateral elements for second order elliptic problems*, M2AN Math. Model. Numer. Anal. **33(4)** (1999), 747–770.
10. T. Ha, J. E. Santos and D. Sheen, *Nonconforming finite element methods for the simulation of waves in viscoelastic solids*, Comput. Methods Appl. Mech. Engrg. **191(49-50)** (2002), 5647–5670.