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Dynamical systems method for solving operator equations *

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Abstract

Consider an operator equation $F(u) = 0$ in a real Hilbert space. The problem of solving this equation is ill-posed if the operator $F'(u)$ is not boundedly invertible, and well-posed otherwise. A general method, dynamical systems method (DSM) for solving linear and nonlinear ill-posed problems in a Hilbert space is presented. This method consists of the construction of a nonlinear dynamical system, that is, a Cauchy problem, which has the following properties: 1) it has a global solution, 2) this solution tends to a limit as time tends to infinity, 3) the limit solves the original linear or nonlinear problem. New convergence and discretization theorems are obtained. Examples of the applications of this approach are given. The method works for a wide range of well-posed problems as well.

1 Introduction

This paper contains a recent development of the theory of DSM (dynamical systems method) earlier developed in papers [2]-[12]. DSM is a general method for solving operator equations, especially nonlinear, ill-posed, but also well-posed operator equations. The author hopes that DSM will demonstrate its practical efficiency and will allow one to solve ill-posed problems which cannot be solved by other methods. This paper is intended for a broad audience: the presentation is simplified considerably, and is non-technical in its present form. Most of the results are presented in a new way. Some of the results and/or proofs are new (Theorems 2.1, 3.1, 3.2, 4.2, 6.2, 7.1, 8.1, Remarks 4.4, 4.5, and the discussion of the stopping rules). We try to emphasize the basic ideas and methods of the proofs.

What is the dynamical systems method (DSM) for solving operator equations?

Consider an equation

$$F(u) := B(u) - f = 0, \quad f \in H, \tag{1.1}$$

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where B is a linear or nonlinear operator in a real Hilbert space H. Some of our results can be generalized to more general spaces, but these generalizations are not discussed here. Throughout the paper we assume that:

$$\sup_{u \in B(u_0,R)} ||F^{(j)}(u)|| \leq M_j, \quad j = 1, 2, \tag{1.2}$$

where $B(u_0, R) := \{u : ||u - u_0|| \leq R\}$, $F^{(j)}(u)$ is the Fréchet derivative, and

$$F(y) = 0, \quad y \in B(u_0, R), \tag{1.3}$$

that is, we assume existence of a solution to (1.1), not necessarily uique globally.

Assumptions (1.2) and (1.3) are our standard assumptions below, unless otherwise stated. Only for well-posed problems in Section 2 we do not assume existence of a solution, but prove it, and sometimes we can assume in these problems $j = 1$ in (1.2), rather than $j = 2$. In all the ill-posed problems we assume existence of the solution to (1.1).

Let $\dot{u}$ denote derivative with respect to time. Consider the dynamical system (the Cauchy problem):

$$\dot{u} = \Phi(t, u), \;\; u(0) = u_0, \tag{1.4}$$

where $\Phi(t, u)$ is locally Lipschitz with respect to $u \in H$ and continuous with respect to $t \geq 0$:

$$\sup_{u,v \in B(u_0,R), t \in [0,T]} ||\Phi(t, u) - \Phi(t, v)|| \leq c||u - v||, \quad c = c(R, u_0, T) > 0. \tag{1.5}$$

One can relax "locally Lipschitz" assumption about Φ (for example, use one-sided inequalities), but we do not discuss this point. Problem (1.4) has a unique local solution if (1.5) holds. The DSM for solving (1.1) consists of solving (1.4), where Φ is so chosen that the following three conditions hold:

$$\exists u(t) \forall t > 0; \quad \exists u(\infty) := \lim_{t \to \infty} u(t); \quad F(u(\infty)) = 0. \tag{1.6}$$

Some of the basic results of this paper are the Theorems which provide the choices of Φ for which (1.6) holds, and the technical tools (Theorems 4.1 and 6.1) basic for our proofs.

Problem (1.1) with noisy data f_δ, $||f_\delta - f|| \leq \delta$, given in place of f, generates the problem:

$$\dot{u}_\delta = \Phi_\delta(t, u_\delta), \;\; u_\delta(0) = u_0, \tag{1.7}$$

The solution u_δ to (1.7), calculated at $t = t_\delta$, will have the property

$$\lim_{\delta \to 0} ||u_\delta(t_\delta) - y|| = 0. \tag{1.8}$$

The choice of t_δ with this property is called the stopping rule. One has usually $\lim_{\delta \to 0} t_\delta = \infty$.

In Section 2 we discuss well-posed problems (1.1), that is, the problems for which

$$\sup_{u\in B(u_0,R)} ||[F'(u)]^{-1}|| \leq m_1, \tag{1.9}$$

and in the other sections ill-posed problems (1.1), for which (1.9) fails, are discussed.

The motivations for this work are:

1) to develop a general method for solving operator equations, especially nonlinear and ill-posed,

and

2) to develop a general approach to constructing convergent iterative schemes for solving these equations.

If (1.6) holds, and if one constructs a convergent discretization scheme for solving Cauchy problem (1.4), then one gets a convergent iterative scheme for solving the original equation (1.1).

2 Well-posed problems

Consider (1.1), let (1.2) hold, and assume

$$(F'(u)\Phi(t,u), F(u)) \leq -g_1(t)||F(u)||^a \quad \forall u \in B(u_0,R), \quad \int_0^\infty g_1 dt = \infty, \tag{2.1}$$

where $g_1 > 0$ is an integrable function, $a > 0$ is a constant. Assume

$$||\Phi(t,u)|| \leq g_2(t)||F(u)||, \quad \forall u \in B(u_0,R), \tag{2.2}$$

where $g_2 > 0$ is such that

$$G(t) := g_2(t)\exp(-\int_0^t g_1 ds) \in L^1(\mathbb{R}_+). \tag{2.3}$$

Remark: Sometimes the assumption (2.2) can be used in the following modified form:

$$||\Phi(t,u)|| \leq g_2(t)||F(u)||^b \quad \forall u \in B, \tag{2.2$'$}$$

where $b > 0$ is a constant. The statement and proof of Theorem 2.1 can be easily adjusted to this assumption.

Our first basic result is the folowing:

Theorem 2.1. *i) If (2.1)-(2.3) hold, and*

$$||F(u_0)|| \int_0^\infty G(t)dt \leq R, \quad a = 2, \tag{2.4}$$

then (1.4) has a global solution, (1.6) holds, (1.1) has a solution $y = u(\infty) \in B(u_0, R)$, *and*

$$||u(t) - y|| \leq ||F(u_0)|| \int_t^\infty G(x)dx, \quad ||F(u(t))|| \leq ||F(u_0)|| \exp(-\int_0^t g_1(x)dx). \tag{2.5}$$

ii) If (2.1)-(2.3) hold, $0 < a < 2$, *and*

$$||F(u_0)|| \int_0^T g_2 ds \leq R, \tag{2.6}$$

where $T > 0$ *is defined by the equation*

$$\int_0^T g_1(s)ds = ||F(u_0)||^{2-a}/(2-a), \tag{2.7}$$

then (1.4) has a global solution, (1.6) holds, (1.1) has a solution $y = u(\infty) \in B(u_0, R)$, *and* $u(t) = y$ *for* $t \geq T$.

iii) If (2.1)-(2.3) hold, $a > 2$, *and*

$$\int_0^\infty g_2(s)h(s)ds \leq R, \tag{2.8}$$

where

$$[||F(u_0)||^{2-a} + (a-2)\int_0^t g_1(s)ds]^{\frac{1}{2-a}} := h(t), \quad \lim_{t\to\infty} h(t) = 0, \tag{2.9}$$

then (1.4) has a global solution, (1.6) holds, (1.1) has a solution $y = u(\infty) \in B(u_0, R)$, *and*

$$||u(t) - u(\infty)|| \leq \int_t^\infty g_2(s)h(s)ds \to 0 \tag{2.10}$$

as $t \to \infty$.

Let us sketch the proof.

Proof of Theorem 2.1. The assumptions about Φ imply local existence and uniqueness of the solution $u(t)$ to (1.4). To prove global existence of u, it is sufficient to prove a uniform with respect to t bound on $||u(t)||$. Indeed, if the maximal interval of the existence of $u(t)$ is finite, say $[0, T)$, and $\Phi(t, u)$ is locally Lipschitz with respect to u, then $||u(t)|| \to \infty$ as $t \to T$.

Assume $a = 2$. Let $g(t) := ||F(u(t))||$. Since H is real, one uses (1.4) and (2.1) to get $g\dot{g} = (F'(u)\dot{u}, F) \leq -g_1(t)g^2$, so $\dot{g} \leq -g_1(t)g$, and integrating one gets the second inequality (2.5), because $g(0) = ||F(u_0)||$. Using (2.2), (1.4) and the second inequality (2.5), one gets:

$$||u(t) - u(s)|| \leq g(0) \int_s^t G(x)dx, \quad G(x) := g_2(x) \exp(-\int_0^x g_1(z)dz). \tag{2.5'}$$

Because $G \in L^1(R_+)$, it follows from (2.5′) that the limit $y := \lim_{t\to\infty} u(t) = u(\infty)$ exists, and $y \in B$ by (2.4). From the second inequality (2.5) and the continuity of F one gets $F(y) = 0$, so y solves (1.1). Taking $t \to \infty$ and setting $s = t$ in (2.5′) yields the first inequality (2.5). The inclusion $u(t) \in B$ for all $t \geq 0$ follows from (2.4) and (2.5′). The first part of Theorem 2.1 is proved. The proof of the other parts is similar. □

There are many applications of this theorem. We mention just a few, and assume that $g_1 = c_1 = const > 0$ and $g_2 = c_2 = const > 0$.

Example 1. Continuous Newton-type method (Gavurin, (1958)):

$\Phi = -[F'(u)]^{-1}F(u)$. Assume that (1.9) holds, then $c_1 = 1, c_2 = m_1$, (2.4) takes the form (*) $m_1(R)||F(u_0)|| \leq R$, and (*) implies that (1.4) has a global solution, (1.6) and (2.5) hold, and (1.1) has a solution in $B(u_0, R)$.

Example 2. Continuous simple iterations method:

Let $\Phi = -F$, and assume $F'(u) \geq c_1(R) > 0$ for all $u \in B(u_0, R)$. Then $c_2 = 1$, $c_1 = c_1(R)$, (2.4) is: $[c_1(R)]^{-1}||F(u_0)|| \leq R$, and the conclusions of Example 1 hold.

Example 3. Continuous gradient method:

Let $\Phi = -[F']^*F$, (1.2) and (1.9) hold, $c_1 = m_1^{-2}$, $c_2 = M_1(R)$, (2.4) is (**) $M_1 m_1^2||F(u_0)|| \leq R$, and (**) implies the conclusions of Example 1.

Example 4. Continuous Gauss-Newton method: Let $\Phi = -([F']^*F')^{-1}[F']^*F$, (1.2) and (1.9) hold, $c_1 = 1$, $c_2 = m_1^2 M_1$, (2.4) is (***) $M_1 m_1^2||F(u_0)|| \leq R$, and (***) implies the conclusions of Example 1.

Example 5. Continuous modified Newton method: Let $\Phi = -[F'(u_0)]^{-1}F(u)$. Assume $||[F'(u_0)]^{-1}|| \leq m_0$, and let (1.2) hold. Then $c_2 = m_0$. Choose $R = (2M_2 m_0)^{-1}$, and $c_1 = 0.5$. Then (2.4) is $2m_0||F(u_0)|| \leq (2M_2 m_0)^{-1}$, that is, $4m_0^2 M_2||F(u_0)|| \leq 1$. Thus, if $4m_0^2 M_2||F(u_0)|| \leq 1$, then the conclusions of Example 1 hold.

Example 6. Descent methods.

Let $\Phi = -\frac{f}{(f',h)}h$, where $f = f(u(t))$ is a differentiable functional $f : H \to [0, \infty)$, and h is an element of H. From (1.4) one gets $\dot{f} = (f', \dot{u}) = -f$. Thus $f = f_0 e^{-t}$, where $f_0 := f(u_0)$. Assume $||\Phi|| \leq c_2|f|^b$, $b > 0$. Then $||\dot{u}|| \leq c_2|f_0|^b e^{-bt}$. Therefore $u(\infty)$ does exist, $f(u(\infty)) = 0$, and $||u(\infty) - u(t)|| \leq ce^{-bt}$, $c = const > 0$.

If $h = f'$, and $f = ||F(u)||^2$, then $f'(u) = 2[F']^*(u)F(u)$, $\Phi = -\frac{f}{||f'||^2}f'$, and (1.4) is a descent method. For this Φ one has $c_1 = \frac{1}{2}$, and $c_2 = \frac{m_1}{2}$, where m_1 is defined in (1.9). Condition (2.4) is: $m_1||F(u_0)|| \leq R$. If this inequality holds, then the conclusions of Example 1 hold.

In Example 6 we have obtained some results from [1]. Our approach is more general than the one in [1], since the choices of f and h do not allow one, for example, to obtain Φ used in Example 5.

3 Linear ill-posed problems

We assume that (1.9) fails. Consider

$$Au = f. \tag{3.1}$$

Let us denote by A) the folowing assumption:

A): A is a linear, bounded operator in H, defined on all of H, the range $R(A)$ is not closed, so (3.1) is an ill-posed problem, there is a y such that $Ay = f$, $y \perp N$, where N is the null-space of A.

Let $B := A^*A$, $q := By = A^*f$, A^* is the adjoint of A. Every solution to (3.1) solves

$$Bu = q, \tag{3.2}$$

and, if $f = Ay$, then every solution to (3.2) solves (3.1). Choose a continuous, monotonically decaying to zero function $\epsilon(t) > 0$, on $\mathbb{R}_+$.

Sometimes it is convenient to assume that

$$\lim_{t\to\infty}(\dot{\epsilon}\epsilon^{-2}) = 0. \tag{3.3}$$

For example, the functions $\epsilon = c_1(c_0 + t)^{-b}$, $0 < b < 1$, where c_0 and c_1 are positive constants, satisfy (3.3). There are many such functions. One can prove ([4], [8]) the following:

Claim: *If $\epsilon(t) > 0$ is a continuous monotonically decaying function on $\mathbb{R}_+$, $\lim_{t\to\infty}\epsilon(t) = 0$, and (3.3) holds, then*

$$\int_0^\infty \epsilon ds = \infty. \tag{3.3$'$}$$

In this Section we do not use assumption (3.3): in the proof of Theorem 3.1 one uses only the monotonicity of a continuous function $\epsilon > 0$ and (3.3$'$). One can drop assumption (3.3$'$), but then convergence is proved in Theorem 3.1 to some element of N, not necessarily to the normal solution y, that is, to the solution orthogonal to N, or, which is the same, to the minimal norm solution to (3.1). However, (3.3) is used (in a slightly weaker form) in Section 4.

Consider problems (1.4) and (1.7) with

$$\Phi := -[Bu + \epsilon(t)u - q], \quad \Phi_\delta = -[Bu_\delta + \epsilon(t)u_\delta - q_\delta], \tag{3.4}$$

where $||q - q_\delta|| \le ||A^*||\delta := C\delta$. Without loss of generality one may assume this $C = 1$, which we do in what follows. Our main result in Sec. 3, is Theorem 3.1, stated below. It yields the following:

Conclusion: *Given noisy data f_δ, every linear ill-posed problem (3.1) under the assumptions A) can be stably solved by the DSM.*

The result presented in Theorem 3.1 is essentially obtained in [8], but our proof is different and much shorter.

Theorem 3.1. *Problem (1.4) with Φ from (3.4) has a unique global solution $u(t)$, (1.6) holds, and $u(\infty) = y$. Problem (1.7) with Φ_δ from (3.4), has a unique global solution $u_\delta(t)$, and there exists t_δ, such that*

$$\lim_{\delta \to 0} ||u_\delta(t_\delta) - y|| = 0. \tag{3.5}$$

This t_δ can be chosen, for example, as a root of the equation

$$\epsilon(t) = \delta^b, \quad b \in (0, 1), \tag{3.6}$$

or of the eqation (3.6'), see below.

Proof of Theorem 3.1. Linear equations (1.4) with bounded operators have unique global solutions. If $\Phi = -[Bu + \epsilon(t)u - q]$, then the solution u to (1.4) is

$$u(t) = h^{-1}(t)U(t)u_0 + h^{-1}(t) \int_0^{||B||} \exp(-t\lambda) \int_0^t e^{s\lambda} h(s) ds \lambda dE_\lambda y, \tag{3.7}$$

where $h(t) := \exp(\int_0^t \epsilon(s)ds) \to \infty$ as $t \to \infty$, E_λ is the resolution of the identity corresponding to the selfadjoint operator B, and $U(t) := e^{-tB}$ is a nonexpansive operator, because $B \geq 0$. Actually, (3.7) can be used also when B is unbounded, $||B|| = \infty$.

Using L'Hospital's rule one checks that

$$\lim_{t \to \infty} \frac{\lambda \int_0^t e^{s\lambda} h(s) ds}{e^{t\lambda} h(t)} = \lim_{t \to \infty} \frac{\lambda e^{t\lambda} h(t)}{\lambda e^{t\lambda} h(t) + e^{t\lambda} h(t) \epsilon(t)} = 1 \quad \forall \lambda > 0, \tag{3.8}$$

provided only that $\epsilon(t) > 0$ and $\lim_{t \to \infty} \epsilon(t) = 0$. From (3.7), (3.8), and the Lebesgue dominated convergence theorem, one gets $u(\infty) = y - Py$, where P is the orthogonal projection operator on the null-space of B. Under our assumptions A), $Py = 0$, so $u(\infty) = y$. If $v(t) := ||u(t) - y||$, then $\lim_{t \to \infty} v(t) = 0$. In general, the rate of convergence of v to zero can be arbitrarily slow for a suitably chosen f. Under an additional a priori assumption on f (for example, the source type assumptions), this rate can be estimated.

Let us describe a method for deriving a stopping rule. One has:

$$||u_\delta(t) - y|| \leq ||u_\delta(t) - u(t)|| + v(t). \tag{3.9}$$

Since $\lim_{t \to \infty} v(t) = 0$, any choice of t_δ such that

$$\lim_{t_\delta \to \infty} ||u_\delta(t_\delta) - u(t_\delta)|| = 0, \tag{3.10}$$

gives a stopping rule: for such t_δ one has $\lim_{\delta \to 0} ||u_\delta(t) - y|| = 0$.

To prove that (3.6) gives such a rule, it is sufficient to check that

$$||u_\delta(t) - u(t)|| \leq \frac{\delta}{\epsilon(t)}. \tag{3.11}$$

Let us prove (3.11). Denote $w := u_\delta - u$. Then

$$\dot{w} = -[Bw + \epsilon w - p], \quad w(0) = 0, \quad ||p|| \leq \delta. \tag{3.12}$$

Integrating (3.12), and using the property $B \geq 0$, one gets (3.11).

Alternatively, multiply (3.12) by w, let $||w|| := g$, use $B \geq 0$, and get $\dot{g} \leq -\epsilon(t)g + \delta$, $g(0) = 0$. Thus, $g(t) \leq \delta \exp(-\int_0^t \epsilon ds) \int_0^t \exp(\int_0^s \epsilon d\tau) ds \leq \frac{\delta}{\epsilon(t)}$. A more precise estimate, also used at the end of the proof of Theorem 3.2 below, yields:

$$||u_\delta(t) - u(t)|| \leq \frac{\delta}{2\sqrt{\epsilon(t)}}, \tag{3.11$'$}$$

and the corresponding stopping time t_δ can be taken as the root of the equation:

$$2\sqrt{\epsilon(t)} = \delta^b, \quad b \in (0, 1). \tag{3.6$'$}$$

Theorem 3.1 is proved. □.

If the rate of decay of v is known, then a more efficient stopping rule can be derived: t_δ is the minimizer of the problem:

$$v(t) + \delta[\epsilon(t)]^{-1} = \min. \tag{3.13}$$

For example, if $v(t) \leq c\epsilon^a(t)$, then t_δ is the root of the equation $\epsilon(t) = (\frac{\delta}{ca})^{\frac{1}{1+a}}$, which one gets from (3.13) with $v = c\epsilon^a$.

One can also use a stopping rule based on an a posteriori choice of the stopping time, for example, the choice by a discrepancy principle.

A method, much more efficient numerically than Theorem 3.1, is given below in Theorem 4.2.

For linear equation (3.2) with exact data this method uses (1.4) with

$$\Phi = -(B + \epsilon(t))^{-1}[Bu + \epsilon(t)u - q] = -u + (B + \epsilon(t))^{-1}q, \tag{3.14}$$

and for noisy data it uses (1.7) with $\Phi_\delta = -u_\delta + (B + \epsilon(t))^{-1}q_\delta$. The linear operator $B \geq 0$ is monotone, so Theorem 4.2 is applicable. For exact data (1.4) with Φ, defined in (3.14), yields:

$$\dot{u} = -u + (B + \epsilon(t))^{-1}q, \quad u(0) = u_0, \tag{3.15}$$

and (1.6) holds if $\epsilon(t) > 0$ is monotone, continuous, decreasing to 0 as $t \to \infty$.

Let us formulate the result:

Theorem 3.2. *Assume A), and let $B := A^*A$, $q := A^*f$. Assume $\epsilon(t) > 0$ to be a continuous, monotonically decaying to zero function on $[0, \infty)$. Then, for any $u_0 \in H$, problem (3.15) has a unique global solution, $\exists u(\infty) = y$, $Ay = f$, and y is the minimal-norm solution to (3.1). If f_δ is given in place of f, $||f - f_\delta|| \leq \delta$, then (3.5) holds, with $u_\delta(t)$ solving (3.15) with q replaced by $q_\delta := A^*f_\delta$, and t_δ is chosen, for example, as the root of (3.6$'$) (or by a discrepancy principle).*

Proof of Theorem 3.2. One has $q = Bz$, where $Az = f$, and the solution to (3.15) is

$$u(t) = u_0 e^{-t} + e^{-t} \int_0^t e^s (B + \epsilon(s))^{-1} B z ds := u_0 e^{-t} + \int_0^{||B||} j(\lambda, t) dE_\lambda z \tag{3.16}$$

where

$$j(\lambda, t) := \int_0^t \frac{\lambda e^s}{[\lambda + \epsilon(s)]e^t} ds, \tag{3.17}$$

and E_λ is the resolution of the identity of the selfadjoint operator B. One has

$$0 \leq j(\lambda, t) \leq 1, \quad \lim_{t \to \infty} j(\lambda, t) = 1 \ \ \lambda > 0, \quad j(0, t) = 0. \tag{3.18}$$

From (3.16)-(3.18) it follows that $\exists u(\infty)$, $u(\infty) = z - P_N z = y$, where y is the minimal-norm solution to (3.1), $N := N(B) = N(A)$ is the null-space of B and of A, and P_N is the orthoprojector onto N in H. This proves the first part of Theorem 3.2.

To prove the second part, denote $w := u_\delta - u$, $g := f_\delta - f$, where we dropped the dependence on δ in w and g for brevity. Then $\dot{w} = -w + (B + \epsilon(t))^{-1} A^* g$, $w(0) = 0$. Thus $w = e^{-t} \int_0^t e^s (B + \epsilon(s))^{-1} A^* g ds$, so $||w|| \leq \delta\, e^{-t} \int_0^t \frac{e^s}{2\sqrt{\epsilon(s)}} ds \leq \frac{\delta}{2\sqrt{\epsilon(t)}}$, where the known estimate (see e.g. [5]) was used: $||(B + \epsilon)^{-1} A^*|| \leq \frac{1}{2\sqrt{\epsilon}}$. Theorem 3.2 is proved. □

4 Nonlinear ill-posed problems with monotone operators

There is a large literature on the equations (1.1) and (1.4) with monotone operators. In the result we present the problem is nonlinear and ill-posed, the new technical tool, Theorem 4.1, is used, and the stopping rules are discussed.

Consider (1.4) with monotone F under standard assumptions (1.2) and (1.3), and

$$\Phi = -A^{-1}_{\epsilon(t)}(u)\big[F(u(t)) + \epsilon(t)(u(t) - \tilde{u}_0)\big], \tag{4.1}$$

where $A = A(u) := F'(u)$, A^* is its adjoint, $\epsilon(t)$ is the same as in Theorem 3.2, and in Theorem 4.2 $\epsilon(t)$ is further specified, $\tilde{u}_0 \in B(u_0, R)$ is an element we can choose to improve the numerical performance of the method. If noisy data are given, then, as in Sec.3, we take

$$F(u) := B(u) - f, \quad \Phi_\delta = -A^{-1}_{\epsilon(t)}(u_\delta)\big[B(u_\delta(t)) - f_\delta + \epsilon(t)(u_\delta(t) - \tilde{u}_0)\big],$$

where $||f_\delta - f|| \leq \delta$, B is a monotone nonlinear operator, $B(y) = f$, and u_δ solves (1.7).

To prove that (1.4) with the above Φ has a global solution and (1.6) holds, we use the following:

Theorem 4.1. *Let $\gamma(t), \sigma(t), \beta(t) \in C[t_0, \infty)$ for some real number t_0. If there exists a positive function $\mu(t) \in C^1[t_0, \infty)$ such that*

$$0 \leq \sigma(t) \leq \frac{\mu(t)}{2}[\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}], \quad \beta(t) \leq \frac{1}{2\mu(t)}[\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}], \quad g_0 \mu(t_0) < 1, \tag{4.2}$$

where g_0 is the initial condition in (4.3), then a nonnegative solution g to the following differential inequality:

$$\dot{g}(t) \leq -\gamma(t)g(t) + \sigma(t)g^2(t) + \beta(t), \quad g(t_0) = g_0, \tag{4.3}$$

satisfies the estimate:

$$0 \leq g(t) \leq \frac{1-\nu(t)}{\mu(t)} < \frac{1}{\mu(t)}, \tag{4.4}$$

for all $t \in [t_0, \infty)$, where

$$0 < \nu(t) = \left(\frac{1}{1-\mu(t_0)g(t_0)} + \frac{1}{2}\int_{t_0}^{t} \left(\gamma(s) - \frac{\dot{\mu}(s)}{\mu(s)} \right) ds \right)^{-1}. \tag{4.5}$$

There are several novel features in this result. First, differential equation, which one gets from (4.3) by replacing the inequality sign by the equality sign, is a Riccati equation, whose solution may blow up in a finite time, in general. Conditions (4.2) guarantee the global existence of the solution to this Riccati equation with the initial condition (4.3). Secondly, this Riccati differential equation cannot be integrated analytically by separation of variables. Thirdly, the coefficient $\sigma(t)$ may grow to infinity as $t \to \infty$, so that the quadratic term does not necessarily has a small coefficient, or the coefficient smaller than $\gamma(t)$. Without loss of generality one may assume $\beta(t) \geq 0$ in Theorem 4.1. The proof of Theorem 4.1 is given in [3].

The main result of this Section is new. It claims a global convergence in the sense that no assumptions on the choice of the initial approximation u_0 are made. Usually one assumes that u_0 is sufficiently close to the solution of (1.1) in order to prove convergence. We take $\tilde{u}_0 = 0$ in Theorem 4.2, because in this theorem $\tilde{u}_0$ does not play any role. The proof is valid for any choice of $\tilde{u}_0$, but then the definition of r in Theorem 4.2 is changed.

Theorem 4.2. *If (1.2) and (1.3) hold, $\tilde{u}_0 = 0$, $R = 3r$, where $r := ||y|| + ||u_0||$, and $y \in N := \{z : F(z) = 0\}$ is the (unique) minimal norm solution to (1.1), then, for any choice of u_0, problem (1.4) with Φ defined in (4.1), $\tilde{u}_0 = 0$, and $\epsilon(t) = c_1(c_0 + t)^{-b}$ with some positive constants c_1, c_0, and $b \in (0,1)$, specified in the proof of Theorem 4.2, has a global solution, this solution stays in the ball $B(u_0, R)$ and (1.6) holds. If $u_\delta(t)$ solves (1.4) with Φ_δ in place of Φ, then there is a t_δ such that $\lim_{\delta \to 0} ||u_\delta(t_\delta) - y|| = 0$.*

Proof of Theorem 4.2. Let us sketch the steps of the proof. Let V solve the equation

$$F(V) + \epsilon(t)V = 0. \tag{4.6}$$

Under our assumptions on F, it is well known that: i) (4.6) has a unique solution for every $t > 0$, and ii) $\sup_{t \geq 0} ||V|| \leq ||y||$, (cf [3]). If F is Fréchet differentiable, then V is differentiable, and $||\dot{V}(t)|| \leq ||y|| |\dot{\epsilon}(t)|/\epsilon(t)$. It is also known that if (1.3) holds, then $\lim_{t \to \infty} ||V(t) - y|| = 0$. We will show that the global solution u to (1.4), with the Φ from (4.1), does exist, and $\lim_{t \to \infty} ||u(t) - V(t)|| = 0$. This is done by deriving a differential inequality for $w := u - V$, and by applying Theorem 4.1 to $g = ||w||$. Since

$||u(t) - y|| \leq ||u(t) - V(t)|| + ||V(t) - y||$, it then follows that (1.6) holds. We also check that $u(t) \in B(u_0, R)$, where $R := 3(||y|| + ||u_0||)$, for any choice of u_0 and a suitable choice of ϵ.

Let us derive the differential inequality for w. One has

$$\dot{w} = -\dot{V} - A^{-1}_{\epsilon(t)}(u)\big[F(u(t)) - F(V(t)) + \epsilon(t)w\big], \tag{4.7}$$

and $F(u) - F(V) = Aw + K$, where $||K|| \leq M_2 g^2/2$, $g := ||w||$ and M_2 is the constant from (1.2). Multiply (4.7) by w, use the monotonicity of F, that is, the property $A \geq 0$, and the estimate $||\dot{V}|| \leq ||y|||\dot{\epsilon}|/\epsilon$, and get:

$$\dot{g} \leq -g + \frac{0.5Mg^2}{\epsilon} + ||y||\frac{|\dot{\epsilon}|}{\epsilon}, \tag{4.8}$$

where $M := M_2$. Inequality (4.8) is of the type (4.3): $\gamma = 1$, $\sigma = 0.5M/\epsilon$, $\beta = ||y||\frac{|\dot{\epsilon}|}{\epsilon}$. Choose

$$\mu(t) = \frac{2M}{\epsilon(t)}. \tag{4.9}$$

Clearly $\mu \to \infty$ as $t \to \infty$. Let us check three conditions (4.2). One has $\frac{\dot{\mu}(t)}{\mu(t)} = \frac{|\dot{\epsilon}|}{\epsilon}$. Take $\epsilon = c_1(c_0 + t)^{-b}$, where $c_j > 0$ are constants, $0 < b < 1$, and choose these constants so that $\frac{|\dot{\epsilon}|}{\epsilon} < \frac{1}{2}$, for example, $\frac{b}{c_0} = \frac{1}{4}$. Then the first condition (4.2) is satisfied. The second condition (4.2) holds if

$$8M||y|||\dot{\epsilon}|\epsilon^{-2} \leq 1. \tag{4.10}$$

One has $\epsilon(0) = c_1 c_0^{-b}$. Choose

$$\epsilon(0) = 4Mr. \tag{4.11}$$

Then

$$|\dot{\epsilon}|\epsilon^{-2} = bc_1^{-1}(c_0 + t)^{b-1} \leq bc_0^{-1}c_1^{-1}c_0^b = \frac{1}{4\epsilon(0)} = \frac{1}{16Mr}, \tag{4.12}$$

so (4.10) holds. Thus, the second condition (4.2) holds. The last condition (4.2) holds because

$$\frac{2M||u_0 - V_0||}{\epsilon(0)} \leq \frac{2Mr}{4Mr} = \frac{1}{2} < 1.$$

By Theorem 4.1 one concludes that $g = ||w(t)|| < \frac{\epsilon(t)}{2M} \to 0$ when $t \to \infty$, and

$$||u(t) - u_0|| \leq g + ||V - u_0|| \leq g(0) + r \leq 3r. \tag{4.13}$$

This estimate implies the global existence of the solution to (1.4), because if $u(t)$ would have a finite maximal interval of existence, $[0, T)$, then $u(t)$ could not stay bounded when $t \to T$, which contradicts the boundedness of $||u(t)||$, and from (4.13) it follows that $||u(t)|| \leq 4r$. We have proved the first part of Theorem 4.2, namely properties (1.6). □

To derive a stopping rule we argue as in Sec.3. One has:

$$||u_\delta(t) - y|| \leq ||u_\delta(t) - V(t)|| + ||V(t) - y||. \tag{4.14}$$

We have already proved that $\lim_{t\to\infty} v(t) := \lim_{t\to\infty} ||V(t) - y|| = 0$. The rate of decay of v can be arbitrarily slow, in general. Additional assumptions, for example, the source-type ones, can be used to estimate the rate of decay of $v(t)$. One derives differential inequality (4.3) for $g_\delta := ||u_\delta(t) - V(t)||$, and estimates g_δ using (4.4). The analog of (4.8) for g_δ contains additional term $\frac{\delta}{\epsilon}$ on the right-hand side. If $\frac{\delta}{\epsilon^2} \leq \frac{1}{16M}$, then conditions (4.2) hold, and $g_\delta < \frac{\epsilon(t)}{2M}$. Let t_δ be the root of the equation $\epsilon^2(t) = 16M\delta$. Then $\lim_{\delta\to 0} t_\delta = \infty$, and (1.8) holds because $||u_\delta(t_\delta) - y|| \leq v(t_\delta) + g_\delta$, $\lim_{t_\delta\to\infty} g_\delta(t_\delta) = 0$ and $\lim_{t_\delta\to\infty} v(t_\delta) = 0$, but the convergence in (1.8) can be slow. See [4] and [5] for the rate of convergence under source assumptions. If the rate of decay of $v(t)$ is known, then one chooses t_δ as the minimizer of the problem, similar to (3.13),

$$v(t) + g_\delta(t) = min, \tag{4.15}$$

where the minimum is taken over $t > 0$ for a fixed small $\delta > 0$. This yields a quasioptimal stopping rule. Theorem 4.2 is proved. □

In [4] a local convergence result, similar to the first part of Theorem 4.1, was obtained, that is, $||u_0 - y||$ was assumed sufficiently small, and no discussion of noisy data was given.

Let us give another result:

Theorem 4.3. *Assume that $\Phi = -F(u) - \epsilon(t)u$, F is monotone, $\epsilon(t)$ as in Theorem 3.2, and (3.3), (1.2) and (1.3) hold. Then (1.6) holds.*

Proof of Theorem 4.3. As in the proof of Theorem 4.2, it is sufficient to prove that $\lim_{t\to\infty} g(t) = 0$, where g, w, and V are the same as in Theorem 4.2, and u solves (1.4) with the Φ defined in Theorem 4.3. Similarly to the derivation of (4.7), one gets:

$$\dot{w} = -\dot{V} - [F(u) - F(V) + \epsilon(t)w]. \tag{4.16}$$

Multiply (4.16) by w, use the monotonicity of F, the estimate $||\dot{V}|| \leq \frac{|\dot{\epsilon}(t)|}{\epsilon(t)}||y||$, which was used also in the proof of Theorem 4.2, and get:

$$\dot{g} \leq -\epsilon(t)g + \frac{|\dot{\epsilon}(t)|}{\epsilon(t)}||y||. \tag{4.17}$$

This implies

$$g(t) \leq e^{-\int_0^t \epsilon(s)ds}[g(0) + \int_0^t e^{\int_0^s \epsilon(x)dx} \frac{|\dot{\epsilon}(s)|}{\epsilon(s)}||y||ds]. \tag{4.18}$$

From our assumptions relation (3.3′) follows, and (4.18) together with (3.3) and (3.3′) imply $\lim_{t\to\infty} g(t) = 0$. Theorem 4.3 is proved. □

Remark 4.4. One can drop assumption (1.2) in Theorem 4.3 and assume only that F is a monotone hemicontinuous operator defined on all of H.

Claim 4.5: *If $\epsilon(t) = \epsilon = const > 0$, then $\lim_{\epsilon\to 0} ||u(t_\epsilon) - y|| = 0$, where $u(t)$ solves (1.4) with $\Phi := -F(u) - \epsilon u$, and t_ϵ is any number such that $\lim_{\epsilon\to 0} \epsilon t_\epsilon = \infty$.*

Proof of the claim. One has $||u(t) - y|| \le ||u(t) - V_\epsilon|| + ||V_\epsilon - y||$, where V_ϵ solves (4.6) with $\epsilon(t) = \epsilon = const > 0$. Under our assumptions on F, equation (4.6) has a unique solution, and $\lim_{\epsilon\to 0} ||V_\epsilon - y|| = 0$. So, to prove the claim, it is sufficient to prove that $\lim_{\epsilon\to 0} ||u(t_\epsilon) - V_\epsilon|| = 0$, provided that $\lim_{\epsilon\to 0} \epsilon t_\epsilon = \infty$. Let $g := ||u(t) - V_\epsilon||$, and $w := u(t) - V_\epsilon$. Because $\dot V_\epsilon = 0$, one has the equation: $\dot w = -[F(u) - F(V_\epsilon) + \epsilon w]$. Multiplying this equation by w, and using the monotonicity of F, one gets $\dot g \le -\epsilon g$, so $g(t) \le g(0)e^{-\epsilon t}$. Therefore $\lim_{\epsilon\to 0} g(t_\epsilon) = 0$, provided that $\lim_{\epsilon\to 0} \epsilon t_\epsilon = \infty$. The claim is proved. □

Remark 4.6. One can prove claims i) and ii), formulated below formula (4.6), using DSM version presented in Theorem 8.1 below.

Claim 4.7: *Assume that F is monotone, (1.2) holds, and $F(y) = 0$. Then claims i) and ii), formulated below formula (4.6), hold.*

Proof. First, note that ii) follows from i) easily, because the assumptions $F(y) = 0$, F is monotone, and $\epsilon > 0$, imply, after multiplying $F(V) - F(y) + \epsilon V = 0$ by $V - y$, the inequality $(V, V - y) \le 0$, from which claim ii) follows. Claim i) follows from Theorem 8.1, proved below. □

Claim 4.8: *Assume that the operator F is monotone, hemicontinuous, defined on all of H, $F(y) = 0$, y is the minimal norm element of $N_F := \{z : F(z) = 0\}$, $\Phi = -F(u) - \epsilon(t)u$, $\epsilon(t) > 0$, is monotone, decaying to zero, and (3.3) holds. Then (1.6) holds for the solution to (1.4).*

Proof. Existence of the unique global solution to (1.4) under our assumptions is known (see e.g. [**?**]) Let $w := u - V_b$, $g := ||w||$, where V_b solves $F(V_b) + bV_b = 0$, $b = const > 0$. It is shown in the proof of Claim 4.7 that $||V_b|| \le ||y||$, and one can prove (see e.g., [2]) that $\lim_{b\to 0} ||V_b - y|| = 0$. One has $||u(t) - y|| \le ||u(t) - V_b|| + ||V_b - y||$. Thus, to prove $\lim_{t\to\infty} ||u(t) - y|| = 0$ it is sufficient to prove that $\lim_{t\to\infty} g(t) = 0$. One has $\dot w = -[F(u) + \epsilon(t)u - F(V_b) - bV_b]$. Multiply this equation by w, use the monotonicity of F and get: $\dot g \le -\epsilon(t)g + |\epsilon(t) - b|||y||$. Denote $h(t) := \exp(\int_0^t \epsilon(s)ds)$. Then,

$$g(\xi) \le g(0)h^{-1}(\xi) + h^{-1}(\xi)\int_0^\xi h(s)|\epsilon(s) - b|ds||y||, \quad \forall b = const \ge 0. \tag{4.19}$$

Clearly $\lim_{\xi\to\infty} g(0)h^{-1}(\xi) = 0$, because $\lim_{t\to\infty} h(t) = \infty$. In fact, $\epsilon^{-1} \le ct + c_0$, where $c_0 := \epsilon^{-1}(0) > 0$, and one can choose $0 < c < 1$ because of (3.3), so $h \ge (ct + c_0)^{\frac{1}{c}}$, and $\lim_{t\to\infty} \epsilon(t)h(t) = \infty$. Choose $b = \epsilon(\xi)$ and apply L'Hôspital's rule to the last term in (4.19). L'Hôspital's rule is applicable, and one gets:

$$\lim_{\xi\to\infty} g(\xi) = \lim_{\xi\to\infty} \frac{|\dot\epsilon(\xi)|}{\epsilon^2(\xi)} \frac{\epsilon(\xi)\int_0^\xi h ds}{h(\xi)} = 0,$$

because $\frac{\epsilon(\xi)\int_0^\xi h ds}{h(\xi)}$ is a bounded function and $\frac{|\dot\epsilon(\xi)|}{\epsilon^2(\xi)} \to 0$ as $\xi \to \infty$. Claim 4.8 is proved. □

The result in claim 4.8 contains the result from [**?**], where additional assumptions are made on $\epsilon(t)$, global existence of the solution to (1.4) is assumed, and the proof contains a gap, because it is not shown that the L'Hôspital's rule can be applied twice.

5 Nonlinear ill-posed problems with non-monotone operators

Assume that $F(u) := B(u) - f$, B is a non-monotone operator, $A := F'(u)$, $\tilde{A} := F'(y)$, $T := A^*A$, $\tilde{T} := \tilde{A}^*\tilde{A}$, $T_\epsilon := T + \epsilon I$, where I is the identity operator, ϵ is as in Theorem 3.2 and $\frac{|\dot{\epsilon}(t)}{\epsilon(t)} < 1$,

$$\Phi := -T_\epsilon^{-1}(u)[A^*(B(u) - f) + \epsilon(u - \tilde{u}_0)], \quad \epsilon = \epsilon(t) > 0, \tag{5.1}$$

and Φ_δ is defined similarly, with f_δ replacing f and u_δ replacing u.

The main result of this Section is:

Theorem 5.1. *If (1.2) and (1.3) hold, $u, u_0 \in B(y, R)$, $y - \tilde{u}_0 = \tilde{T}z$, $||z|| << 1$, (that is, $||z||$ is sufficiently small), and R is sufficiently small, then problem (1.4) has a unique global solution and (1.6) holds. If $u_\delta(t)$ solves (1.7), then there exists a t_δ such that $\lim_{\delta \to 0} ||u_\delta(t_\delta) - y|| = 0$.*

The derivation of the stopping rule, that is, the choice of t_δ, is based on the ideas presented in Sec.4 (cf [8], [5]).

Sketch of proof of Theorem 5.1. Proof of Theorem 5.1 consists of the following steps.

First we prove that $g := ||w|| := ||u(t) - y||$ satisfies a differential inequality (4.3), and, applying (4.4), conclude that $g(t) < \mu^{-1}(t) \to 0$ as $t \to \infty$. A new point in this derivation (compared with the one for monotone operators) is the usage of the source assumption $y - u_0 = \tilde{T}z$.

Secondly, we derive the stopping rule using the ideas from Sec.4. The source assumption allows one to get a rate of convergence (see [2] and [5]). Details of the proof are technical and are not included. One can see [5] for some proofs.

Let us sketch the derivation of the differential inequality for g. Write $B(u) - f = B(u) - B(y) = Aw + K$, where $||K|| \leq \frac{M_2}{2} g^2$, and $\epsilon(u - \tilde{u}_0) = \epsilon w + \epsilon(y - \tilde{u}_0) = \epsilon w + \epsilon \tilde{T} z$. Then (5.1) can be written as

$$\Phi = -w - T_\epsilon^{-1} A^* K - \epsilon T_\epsilon^{-1} \tilde{T} z, \quad \epsilon := \epsilon(t). \tag{5.2}$$

Multiplying (1.4), with Φ defined in (5.2), by w, one gets:

$$g\dot{g} \leq -g^2 + \frac{M_2}{2} ||T_{\epsilon(t)}^{-1} A^*|| g^3 + \epsilon(t) ||T_{\epsilon(t)}^{-1} \tilde{T}|| ||z|| g. \tag{5.3}$$

Since $g \geq 0$, one obtains:

$$\dot{g} \leq -g + \frac{M_2}{4\sqrt{\epsilon(t)}} g^2 + \epsilon(t) ||T_\epsilon^{-1} \tilde{T}|| ||z||, \tag{5.4}$$

where the estimate $||T_\epsilon^{-1}A^*|| \leq \frac{1}{2\sqrt{\epsilon}}$ was used. Clearly,

$$||T_\epsilon^{-1}\tilde{T}|| \leq ||(T_\epsilon^{-1} - \tilde{T}_\epsilon^{-1})\tilde{T}|| + ||\tilde{T}_\epsilon^{-1}\tilde{T}||, \quad ||\tilde{T}_\epsilon^{-1}\tilde{T}|| \leq 1, \quad \epsilon||T_\epsilon^{-1}|| \leq 1,$$

and

$$T_\epsilon^{-1} - \tilde{T}_\epsilon^{-1} = T_\epsilon^{-1}(A^*A - \tilde{A}^*\tilde{A})\tilde{T}_\epsilon^{-1}.$$

One has:

$$||A^*A - \tilde{A}^*\tilde{A}|| \leq 2M_2M_1g, \quad ||z|| << 1.$$

Let $2M_1M_2||z|| \leq \frac{1}{2}$. This is possible since $||z|| << 1$. Using the above estimates, one transforms (5.4) into the following inequality:

$$\dot{g} \leq -\frac{1}{2}g + \frac{M_2}{4\sqrt{\epsilon(t)}}g^2 + ||z||\epsilon. \tag{5.5}$$

Now, apply Theorem 4.1 to (5.5), choosing

$$\mu = \frac{2M_2}{\sqrt{\epsilon}}, \quad \frac{|\dot{\epsilon}|}{\epsilon} < \frac{1}{2}, \quad 16M_2||z||\sqrt{\epsilon(0)} < 1, \quad \textit{and} \quad \frac{2M_2||u_0 - y||}{\sqrt{\epsilon(0)}} < 1.$$

Then conditions (4.2) are satisfied, and Theorem 4.1 yields the estimate:

$$g(t) < \frac{\sqrt{\epsilon(t)}}{2M_2}.$$

This is the main part of the proof of Theorem 5.1. □

6 Nonlinear ill-posed problems: avoiding inverting of operators in the Newton-type continuous schemes

In the Newton-type methods for solving well-posed nonlinear problems, for example, in the continuous Newton method (1.4) with $\Phi = -[F'(u)]^{-1}F(u)$, the difficult and expensive part of the solution is inverting the operator $F'(u)$. In this section we give a method to avoid inverting of this operator. This is especially important in the ill-posed problems, where one has to invert some regularized versions of F', and to face more difficulties than in the well-posed problems.

Consider problem (1.1) and assume (1.2), (1.3) and (1.9). Thus, we discuss our method in the simplest well-posed case.

Replace (1.4) by the following Cauchy problem (dynamical system):

$$\dot{u} = -QF, \quad u(0) = u_0, \tag{6.1}$$

$$\dot{Q} = -TQ + A^*, \quad Q(0) = Q_0, \tag{6.2}$$

where $A := F'(u)$, $T := A^*A$, and $Q = Q(t)$ is a bounded operator in H.

First let us state our new technical tool: an operator version of the Gronwall inequality (cf [9]).

Theorem 6.1. *Let*

$$\dot{Q} = -T(t)Q(t) + G(t), \quad Q(0) = Q_0, \tag{6.3}$$

where $T(t)$, $G(t)$, and $Q(t)$ are linear bounded operators on a real Hilbert space H. If there exists $\epsilon(t) > 0$ such that

$$(T(t)h, h) \geq \epsilon(t)||h||^2 \quad \forall h \in H, \tag{6.4}$$

then

$$||Q(t)|| \leq e^{-\int_0^t \epsilon(s)ds}[||Q(0)|| + \int_0^t ||G(s)||e^{\int_0^s \epsilon(x)dx}\, ds]. \tag{6.5}$$

Let us turn now to a proof of Theorem 6.2, formulated at the end of this Section. This theorem is the main result of Section 6.

Applying (6.5) to (6.2), and using (1.2) and (1.9), which implies

$$(T(t)h, h) \geq c||h||^2 \quad \forall h \in H, \quad c = const > 0, \tag{6.6}$$

one gets:

$$||Q(t)|| \leq e^{-ct}[||Q(0)|| + \int_0^t M_1 e^{cs}\, ds] \leq [||Q_0|| + M_1 c^{-1}] := c_1, \tag{6.7}$$

as long as $u(t) \in B(u_0, R)$.

Let $u(t) - y := w$, $||w|| := g$, $\tilde{A} := F'(y)$. Since $F(y) = 0$, one has $F(u) = \tilde{A}w + K$, where $||K|| \leq 0.5M_2 g^2 := c_0 g^2$, and M_2 is the constant from (1.2). Rewrite (6.1) as

$$\dot{w} = -Q[\tilde{A}w + K]. \tag{6.8}$$

Let $\Lambda := I - Q\tilde{A}$. Multiply (6.8) by w and get

$$g\dot{g} \leq -g^2 + (\Lambda w, w) + c_0 g^3, \quad c_0 = const > 0. \tag{6.9}$$

We prove below that

$$\sup_{t \geq 0} ||\Lambda|| \leq \lambda < 1. \tag{6.10}$$

From (6.9) and (6.10) one gets the following differential inequality:

$$\dot{g} \leq -\gamma g + c_0 g^2, \quad 0 < \gamma < 1, \ \gamma := 1 - \lambda, \tag{6.11}$$

which implies:

$$g(t) \leq re^{-\gamma t}, \quad r := g(0)[1 - g(0)c_0]^{-1}, \tag{6.12}$$

provided that

$$g(0)c_0 < 1. \tag{6.13}$$

Inequality (6.13) holds if u_0 is sufficiently close to y.

From (6.12) and (6.11) it folows that $u(\infty) = y$. Thus, (1.6) holds.

The trajectory $u(t) \in B(u_0, R)$, $\forall t > 0$, provided that

$$\int_0^\infty ||\dot{u}||dt = \int_0^\infty ||\dot{w}||dt \leq r + \frac{c_0 r^2}{2\gamma} \leq R. \tag{6.14}$$

This inequality holds if u_0 is sufficiently close to y, that is, r is sufficiently small.

To complete the argument, let us prove (6.10). One has:

$$\dot{\Lambda} = -\dot{Q}\tilde{A} = -T\Lambda + A^*(A - \tilde{A}). \tag{6.15}$$

One has $||A - \tilde{A}|| \leq M_2 g$. Using (6.12) and Theorem 6.1, one gets

$$||\Lambda|| \leq e^{-ct}[||\Lambda_0|| + rM_1M_2 \int_0^t e^{(c-\gamma)s}ds]. \tag{6.16}$$

Thus,

$$||\Lambda|| \leq ||\Lambda_0|| + Cr := \lambda, \quad C := M_1M_2 \sup_{t>0} \frac{e^{-\gamma t} - e^{-ct}}{c - \gamma}. \tag{6.17}$$

If u_0 is sufficiently close to y and Q_0 is sufficiently close to $\tilde{A}^{-1}$, then $\lambda > 0$ can be made arbitrarily small. We have proved:

Theorem 6.2. *If (1.2), (1.3) and (1.9) hold, Q_0 and u_0 are sufficiently close to $\tilde{A}^{-1}$ and y, respectively, then problem (6.1)-(6.2) has a unique global solution, (1.6) holds, and $u(t)$ converges to y, which solves (1.1), exponentially fast.*

In [9] a generalization of Theorem 6.2 is given for ill-posed problems.

7 Iterative schemes

In this section we present a method for constructing convergent iterative schemes for a wide class of well-posed equations (1.1). Some methods for constructing convergent iterative schemes for a wide class of ill-posed problems are given in [3]. There is an enormous literature on iterative methods.

Consider a discretization scheme for solving (1.4) with $\Phi = \Phi(u)$, so that we assume no explicit time dependence in Φ:

$$u_{n+1} = u_n + h\Phi(u_n), \quad u_0 = u_0, \quad h = const > 0. \tag{7.1}$$

One of our results from [3], concerning the well-posed equations (1.1) is Theorem 7.1, formulated below. Its proof is shorter and simpler than in [3].

Theorem 7.1. *Assume (1.2), (1.3), (1.9), (2.1)-(2.4) with $a = 2$, $g_1 = c_1 = const > 0$, $g_2 = c_2 = const > 0$, $||\Phi'(u)|| \leq L_1$, for $u \in B(y, R)$. Then, if $h > 0$ is sufficiently small, and u_0 is sufficiently close to y, then (7.1) produces a sequence u_n for which*

$$||u_n - y|| \leq Re^{-chn}, \quad ||F(u_n)|| \leq ||F_0||e^{-chn}, \tag{7.2}$$

where $R := \frac{c_2||F_0||}{c_1}$, $F_0 = F(u_0)$, $c = const > 0$, $c < c_1$.

Proof of Theorem 7.1. The proof is by induction. For $n = 0$ estimates (7.2) are clear. Assuming these estimates for $j \leq n$, let us prove them for $j = n + 1$. Let $F_n := F(u_n)$, and let $w_{n+1}(t)$ solve problem (1.4) on the interval (t_n, t_{n+1}), $t_n := nh$, with $w_{n+1}(t_n) = u_n$. By (2.5) (with $G = c_2e^{-c_1t}$) and (7.2) one gets:

$$||w_{n+1}(t) - y|| \leq \frac{c_2}{c_1}||F_n||e^{-c_1t} \leq Re^{-cnh-c_1t}, \quad t_n \leq t \leq t_{n+1}. \tag{7.3}$$

One has:

$$||u_{n+1} - y|| \leq ||u_{n+1} - w_{n+1}(t_{n+1})|| + ||w_{n+1}(t_{n+1}) - y||, \tag{7.4}$$

and

$$||u_{n+1} - w_{n+1}(t_{n+1})|| \leq \int_{t_n}^{t_{n+1}} ||\Phi(u_n) - \Phi(w_{n+1}(s))||ds$$

$$\leq L_1c_2h \int_{t_n}^{t_{n+1}} ||F(w_{n+1}(t))||dt \leq L_1c_1h^2Re^{-cnh}, \tag{7.5}$$

where we have used the formula $R := \frac{c_2||F_0||}{c_1}$, and the estimate:

$$||F(w_{n+1}(t))|| \leq ||F_n||e^{-c_1(t-t_n)} \leq ||F_0||e^{-cnh-c_1(t-t_n)}. \tag{7.6}$$

From (7.3)-(7.6) it follows that:

$$||u_{n+1} - y|| \leq Re^{-cnh}(e^{-c_1h} + c_1L_1h^2) \leq Re^{-c(n+1)h}, \tag{7.7}$$

provided that

$$e^{-c_1h} + c_1L_1h^2 \leq e^{-ch}. \tag{7.8}$$

Inequality (7.8) holds if h is sufficiently small and $c < c_1$. So, the first inequality (7.2), with $n + 1$ in place of n, is proved if h is sufficiently small and $c < c_1$.

Now

$$||F(u_{n+1})|| \leq ||F(u_{n+1}) - F(w_{n+1}(t))|| + ||F(w_{n+1}(t))||, \quad t_n \leq t \leq t_{n+1}. \tag{7.9}$$

Using (1.2) and (7.5), one gets:

$$||F(u_{n+1}) - F(w_{n+1}(t_{n+1}))|| \leq M_1||u_{n+1} - w_{n+1}(t_{n+1})|| \leq M_1c_2L_1h^2||F_0||e^{-cnh}. \tag{7.10}$$

From (7.9) and (7.10) it follows that:

$$||F(u_{n+1})|| \leq ||F_0||e^{-cnh}(e^{-c_1 h} + M_1 c_2 L_1 h^2) \leq ||F_0||e^{-c(n+1)h}, \tag{7.11}$$

provided that

$$e^{-c_1 h} + M_1 c_2 L_1 h^2 \leq e^{-ch}. \tag{7.12}$$

Inequality (7.12) holds if h is sufficiently small and $c < c_1$. So, the second inequality (7.2) with $n+1$ in place of n is proved if h is sufficiently small and $c < c_1$. Theorem 7.1 is proved. □

In the well-posed case, if $F(y) = 0$, the discrete Newton's method

$$u_{n+1} = u_n - [F'(u_n)]^{-1}F(u_n),\ u_0 = u(0),$$

converges superexponentially if u_0 is sufficiently close to y. Indeed, if $v_n := u_n - y$, then $v_{n+1} = v_n - [F'(u_n)]^{-1}[F'(u_n)v_n + K]$ where $||K|| \leq \frac{M_2}{2}||v_n||^2$. Thus, $g_n := ||v_n||$ satisfies the inequality: $g_{n+1} \leq qg_n^2$, where $q := \frac{m_1 M_2}{2}$. Therefore $g_n \leq q^{2^n-1}g_0^{2^n}$, and if $0 < qg_0 < 1$, then the method converges superexponentially.

If one uses the iterative method $u_{n+1} = u_n - h[F'(u_n)]^{-1}F(u_n)$, with $h \neq 1$, then, in the well-posed case, assuming that this method converges, it converges exponentially, that is, slower than in the case $h = 1$.

The continuous analog of the above method

$$\dot{u} = -a[F'(u)]^{-1}F(u), \quad u(0) = u_0,$$

where $a = const > 0$, converges at the rate $O(e^{-at})$. Indeed, if $g(t) := ||F(u(t))||$, then $g\dot{g} = -ag^2$, so $g(t) = g_0 e^{-at}$, $||\dot{u}|| \leq am_1 g_0 e^{-at}$. Thus

$$||u(t) - u(\infty)|| \leq m_1 g_0 e^{-at}, \quad \textit{and} \quad F(u(\infty)) = 0.$$

In the continuous case one does not have superexponential convergence no matter what $a > 0$ is (see [11]).

8 A spectral assumption

In this section we introduce the spectral assumption which allows one to treat some nonlinear non-monotone operators.

Assumption S: *The set* $\{r, \varphi : \pi - \varphi_0 < \varphi < \pi + \varphi_0,\ \varphi_0 > 0, 0 < r < r_0\}$, *where* φ_0 *and* r_0 *are arbitrarily small, fixed numbers, consists of the regular points of the operator* $A := F'(u)$ *for all* $u \in B(u_0, R)$.

Assumption S implies the estimate:

$$||(F'(u) + \epsilon)^{-1}|| \leq \frac{1}{\epsilon \sin\varphi_0}, \quad \epsilon < r_0(1 - \sin\varphi_0), \quad \epsilon = const > 0, \tag{8.1}$$

because $||(A-z)^{-1}|| \leq \frac{1}{dist(z,s(A))}$, where $s(A)$ is the spectrum of a linear operator A, and $dist(z, s(A))$ is the distance from a point z of a complex plane to the spectrum. In our case, $z = -\epsilon$, and $dist(z, s(A)) = \epsilon \sin\varphi_0$, if $\epsilon < r_0(1 - \sin\varphi_0)$.

Theorem 8.1. *If (1.2) and (8.1) hold, and* $0 < \epsilon < r_0(1-\sin\varphi_0)$*, then problem (4.6), with* $\epsilon(t) = \epsilon = const > 0$*, is solvable, problem (1.4), with* Φ *defined in (4.1) and* $\tilde{u}_0 = 0$*, has a unique global solution,* $\exists u(\infty)$*, and* $F(u(\infty)) + \epsilon u(\infty) = 0$*. Every solution to the equation* $F(V) + \epsilon V = 0$ *is isolated.*

Proof of Theorem 8.1. Let $g = g(t) := ||F(u(t)) + \epsilon u(t)||$, where $u = u(t)$ solves locally (1.4), where Φ is defined in (4.1) and $\tilde{u}_0 = 0$. Then:

$$g\dot{g} = -((F'(u) + \epsilon)(F'(u) + \epsilon)^{-1}(F(u) + \epsilon u), F(u) + \epsilon u) = -g^2, \tag{8.2}$$

so

$$g = g_0 e^{-t},\ g_0 := g(0); \quad ||\dot{u}|| \leq \frac{g_0}{\epsilon \sin\varphi_0} e^{-t}. \tag{8.3}$$

Thus,

$$||u(t) - u(\infty)|| \leq \frac{g_0}{\epsilon \sin\varphi_0} e^{-t}, \quad ||u(t) - u_0|| \leq \frac{g_0}{\epsilon \sin\varphi_0}, \quad F(u(\infty)) + \epsilon u(\infty) = 0. \tag{8.4}$$

Therefore equation

$$F(V) + \epsilon V = 0, \quad \epsilon = const > 0, \tag{8.5}$$

has a solution in $B(u_0, R)$, where $R = \frac{g_0}{\epsilon \sin\varphi_0}$.

Every solution to equation (8.5) is isolated. Indeed, if $F(W)+\epsilon W = 0$, and $\psi := V - W$, then $F(V) - F(W) + \epsilon\psi = 0$, so $[F'(V) + \epsilon]\psi + K = 0$, where $||K|| \leq \frac{M_2}{2}||\psi||^2$. Thus, using (8.1), one gets $||\psi|| \geq \frac{2\epsilon \sin\varphi_0}{M_2}$. Consequently, if $||\psi||$ is sufficiently small, then $\psi = 0$. Theorem 8.1 is proved. □

Assumption S was introduced by the author and applied to deconvolution problems in [**?**]

References

[1] Alber, Ya., Continuous Newton-type processes, Diff Uravneniya, 7, N1, (1971), 1931-1945. (in Russian)

[2] Airapetyan, R., Ramm, A.G., Smirnova, A. B., Continuous analog of Gauss-Newton method, Math. Models and Methods in Appl. Sci., 9, N3, (1999), 463-474.

[3] Airapetyan, R., Ramm, A.G., Dynamical systems and discrete methods for solving nonlinear ill-posed problems, Appl.Math.Reviews, vol. 1, Ed. G. Anastassiou, World Sci. Publishers, 2000, pp.491-536.

[4] Airapetyan, R., Ramm, A.G., Smirnova, A. B., Continuous methods for solving nonlinear ill-posed problems, in the book "Operator theory and applications", Amer. Math. Soc., Fields Institute Communications, Providence,RI, 2000, pp. 111-138. (editors A.G.Ramm, P.N.Shivakumar, A.V.Strauss).

[5] B.Kaltenbacher, A.Neubauer, Ramm, A.G., Convergence rates of the continuous regularized Gauss-Newton method, Jour. Inv. Ill-Posed Probl., 10, N3, (2002), 261-280.

[6] Ramm, A.G., Smirnova, A. B., A numerical method for solving nonlinear ill-posed problems, Nonlinear Funct. Anal. and Optimiz., 20, N3, (1999), 317-332.

[7] Ramm, A.G., Smirnova, A. B., On stable numerical differentiation, Mathem. of Computation, 70, (2001), 1131-1153.

[8] Ramm, A.G., Linear ill-posed problems and dynamical systems, Jour. Math. Anal. Appl., 258, N1, (2001), 448-456.

[9] Ramm, A.G., Smirnova, A. B., Continuous regularized Gauss-Newton-type algorithm for nonlinear ill-posed equations with simultaneous updates of inverse derivative, Intern. Jour. of Pure and Appl Math., 2, N1, (2002), 23-34.

[10] Ramm, A.G., A numerical method for some nonlinear problems, Math. Models and Meth. in Appl.Sci., 9, N2, (1999), 325-335.

[11] Ramm, A.G., Acceleration of convergence of a continuous analog of the Newton method, Applic. Analysis, 81, N4, (2002), 1001-1004.

[12] Ramm, A.G., Regularization of ill-posed problems with unbounded operators, J. Math. Anal. Appl., 271, (2002), 547-550.

[13] Ramm, A.G., On equations of the first kind, Diff. eq., 4, (1968), 2056-2060.

[14] Ramm, A.G., Stable solutions of some ill-posed problems, Math. Meth. in the appl. Sci., 3, (1981), 336-363.

[15] K.Deimling, Nonlinear functional analysis, Springer Verlag, Berlin, 1985, p.99.

[16] Alber, Ya. and Ryasantseva, I., On regularized evolution equations, Funct. Diff. Eqs, 7, (2000), 177-187.

[17] Ramm, A.G., Galstian, A., On deconvolution methods, Intern. Jour. of Engin. Sci., 41, N1, (2003), 31-43.