Remarks on 3-folds in $\mathbb{P}^5$ & quadruple point formula.

- **Introduction**

$$X^n \subset \mathbb{P}^n$$

where $X$ is nondeg irreducible.

$$\Sigma_k(X) = \text{The subvariety of } G(1,N) \text{ consisting of all } k\text{-secant lines of } X.$$  

$$S_k(X) = \bigcup_{l \in \Sigma_k(X)} l \subset \mathbb{P}^n$$

So, we have descending filtrations.

- $$G(1,N) = \Sigma_0(X) \supseteq \Sigma_1(X) \supseteq \cdots \supseteq \Sigma_t(X) = \cdots \Sigma_\infty(X)$$

  where $\Sigma_\infty(X)$ is the set of all lines contained in $X$ (Fano scheme of $X$).

- $$\mathbb{P}^n = S_0(X) = S_1(X) \supseteq S_2(X) \supseteq \cdots \supseteq S_t(X) = \cdots = S_\infty(X)$$

  where $S_\infty(X)$ is the subvariety swept out by lines contained in $X$.

**Remarks**

1. $$\min \{ t \mid \Sigma_t(X) = \Sigma_\infty(X) \text{ or } S_t(X) = S_\infty(X) \}$$

   $$\leq \deg(X) - \text{codim}(X) + 2.$$  

   i.e., $\# \left[ \deg(X) - \text{codim}(X) + 2 \right]$-secant line of $X$ by Bertini’s theorem.

   * $\left[ \deg(X) - \text{codim}(X) + 1 \right]$-secant line is called ‘extremal’
2. If $X \subset \mathbb{P}^F$ is a hypersurface of degree $m$, then $S_k(X) \subset \mathbb{P}^F \subset \mathbb{P}^N$, $\forall k > m$.

3. If $X$ is smooth of dim $n$, then, by Z. Ran (1991)
   
   $S_{n+2}(X) \not\subset \mathbb{P}^F$
   
   $\dim S_{n+2}(X) \leq n+1$

   "(dim + 2)-secant lemma"
   
   Which generalized the classical trisecant lemma for space curves.

From now on, consider codim 2 projective subvarieties.

Then,

$\mathbb{P}^{n+2} = S_2(X) \supseteq S_3(X) \supseteq \cdots \supseteq S_{n+2}(X) \supseteq \cdots$

- $\dim X = 1$, $X \subset \mathbb{P}^3$

   By trisecant lemma,
   
   $\dim \Sigma_3(X) = 1$ unless $C$ is a twisted cubic or $C$ is a c.i of two quadrics.

So

$\mathbb{P}^3 = S_2(X) \supseteq S_3(X)$: Singular surface in $\mathbb{P}^3$. 
• $X^n \subset \mathbb{P}_\mathbb{C}^{n+2}$: nondegenerate & integral.
  
  If $X \subset \mathbb{P}_\mathbb{C}^{m} \subset \mathbb{P}_\mathbb{C}^{n+2}$
  
  then hypersurface of degree $m$
  
  $S_{m+1}(X) \subset \mathbb{F} \subset \mathbb{P}_\mathbb{C}^{n+2}$
  
  (trivial !!)
  
  However, is the converse true?
  
  i.e., if $S_{m+1}(X) = \mathbb{P}_\mathbb{C}^{n+2}$, $m \leq n = \dim(X)$
  
  then $\exists$? a hypersurface $F_i$ of
  
  degree $m$ which contains $X$.
  
  It would be desirable to avoid assumption
  concerning singularities of $X$.
  
  • F. Severi (1901)
  
  If $S_3(X) = \mathbb{P}^3$ for a smooth surface $X \subset \mathbb{P}^4$
  
  then either $X \subset \mathbb{Q}$: a quadric hypersurface
  
  or $S_3(X) = \bigcup_{\lambda \in \mathbb{C}} \mathbb{P}_\lambda^2$, $D_\lambda \subset \mathbb{P}_\lambda^2$ is
  
  a plane curve of degree $\geq 3$, i.e.
  
  $D_\lambda = X \cap \mathbb{P}_\lambda$, $X = \bigcup_{\lambda \in \mathbb{C}} D_\lambda$
Note that the quintic elliptic scroll is not contained in any quadric & cut out by cubics, but \( S_3(X) = \mathbb{P}^4 \).

A. Aure (1988) showed that

\[ S_3(X) = \mathbb{P}^4 \]

\( \iff \) Either \( X \subset \text{Quadric} \) (trivial case!!)

or the elliptic quintic scroll

(by using the genus bound for space curves due to Gruson-Peskine)

Note that for all smooth surfaces \( X \subset \mathbb{P}^4 \),

\[ S_4(X) = \mathbb{P}^4 \]

by Z. Ran’s lemma.

Rmk: The elliptic quintic scroll in \( \mathbb{P}^4 \) can be defined as the degeneracy locus as follows:

\[
0 \rightarrow \bigoplus \mathcal{O}_{\mathbb{P}^4} \rightarrow \Omega^2_{\mathbb{P}^4}(3) \rightarrow \mathcal{G}^0_X(3) \rightarrow 0
\]

\[ I_S = \langle F_i : i \in I, \rangle \]

\[ \deg F_i = 3. \]
• let $X^m \subseteq \mathbb{P}^{n+2}$ (m > 2) be a l.c.i &
subcanonical subvariety.

**Thm (Z. Ran) (1983)**

For $m \leq n$,
\[ S_{m+1}(X) = \mathbb{P}^{n+2} \]
\[ \iff \exists F: \text{hypersurface of degree } m \text{ which contains } X \]
\[ \Rightarrow X \text{ is a c.i.} \]

(proof of idea)

Since $X$ is a subcanonical subvariety of $\mathbb{P}^{n+2}$, $X$ is a zero locus of a section $s \in H^0(\mathcal{E})$, $\mathcal{E} : \text{rk2 vector bundle}$.

\[ 0 \to \mathcal{O}_{\mathbb{P}^{n+2}} \xrightarrow{s} \mathcal{E} \xrightarrow{g_x} g_x(c_1(\mathcal{E})) \to 0 \]

Let $e(k) = C_2(\mathcal{E}(-k))$

**Key!!** If $S_{m+1}(X) = \mathbb{P}^{n+2}$ then $e(k) = 0$ for some $k \leq m$

\[ H^0(\mathcal{E}_x(k)) = 0 \]

(This is a nice connection between enumerative geometry & vector bundle technique)
What can be said about smooth 3-folds of codim 2 along this line?

\[ P^5_\phi = S_2(X) \supseteq S_3(X) \supseteq S_4(X) \supseteq S_5(X) = P^5_\phi \]

Note that \( X \) is not always subcanonical.

Our goal is the following:

1. \( S_3(X) = P^5_\phi \)
   \[ \iff X \subset Q \text{ (trivial case) or \{finite lists of nontrivial examples\}} \]

2. \( S_4(X) = P^5_\phi \)
   \[ \iff X \subset \text{a cubic hypersurface or \{nontrivial examples\}} \]

\[\star\] Our question can be interpreted via a generic projection \( \Pi_p : X \to X' \leq P^4 \).
A generic projection $\pi_p : X \rightarrow Y \subseteq \mathbb{P}^4$, where $p = (0, 0, 0, 0, 0, 1)$ & the monoidal construction for a 3-fold $X \subseteq \mathbb{P}^5$

$X_4 \subseteq X_3 \subseteq X_2 \subseteq X_1 = X \subseteq \mathbb{P}^5$

$\pi_p : \pi^{-1}_p(Y_i) = \pi^{-1}_p(Y_{i+1})$

$Y_4 \subseteq Y_3 \subseteq Y_2 \subseteq Y_1 = Y \subseteq \mathbb{P}^4 \Rightarrow [x_0, x_1, \ldots, x_4]$

where $Y_i = \{ y \in Y = \pi_p(X) | \pi^{-1}_p(y) \geq i \}$

$X_i = \pi^{-1}_p(Y_i)$

Facts: (J. Mather, J. Roberts, Ran)

1. $\dim Y_i \leq 4 - i$
2. $Y_5 = \emptyset$

Observation:

- $S_4(X) = \mathbb{P}^5$ if and only if $|Y_4| = \text{the \# of all quadrisection lines through } p = (0, 0, 0, 0, 0, 1)$ is empty (zero)

- $S_3(X) = \mathbb{P}^5 \iff Y_3 = \emptyset$
• let \( b_4(x) = |Y_4| \) be the \# of all quadriseant lines through a general point \( p = (0, 0, 0, 0, 0, 1) \).
  Then, we can calculate \( b_4(x) \) in terms of \( \chi(\mathcal{O}_x), \chi(\mathcal{O}_{x \cap H}), \pi_1, \deg(x) \) by using monoidal construction & the Giambelli-Thom-Porteous formula.

We have a short exact sequence:

\[
0 \rightarrow \mathcal{E}(\mathcal{O}) \rightarrow \mathcal{O}_{\mathbb{P}^4}^{\oplus 5} \oplus \mathcal{O}_{\mathbb{P}^4}^{(2)} \oplus \mathcal{O}_{\mathbb{P}^4}^{(3)} \rightarrow \pi_{1*} \mathcal{O}_x(3) \rightarrow 0
\]

\[
p = (0, 0, 0, 0, 0, 1) = \mathbb{Z}(T_0, T_1, T_2, T_3, T_4)
\]

\[
\begin{aligned}
\text{why?} & \quad \text{length} \left( \pi_{1*}^{-1}(y) \right) \leq 4 \\
\Rightarrow & \quad \pi_{1*}^{-1}(y) \subset \overline{py} \cong \mathbb{P}^1 \\
\text{is always 3-normal}
\end{aligned}
\]


Lemma

(1) \[ 0 \to \mathcal{E}(3) \to \mathcal{O}_{P^4} \otimes \mathcal{O}(1) \oplus \mathcal{O}(2) \oplus \mathcal{O}(3) \to \pi_{P^4} \mathcal{O}(3) \to 0 \]

Claim: \[ y \in Y_3 \iff \varphi_3: \mathcal{E}(3) \to \mathcal{O}_{P^4} \otimes \mathcal{O}(1) \]

is surjective

i.e. \[ Y_3 = \{ y \in Y \mid \text{rank}(\varphi_3 \otimes k(y)) \leq 1 \} \]

(2) \[ 0 \to \mathcal{E}(3) \to \mathcal{O}_{P^4} \otimes \mathcal{O}(1) \oplus \mathcal{O}(2) \oplus \mathcal{O}(3) \to \pi_{P^4} \mathcal{O}(3) \to 0 \]

\[ Y_4 = \{ y \in Y \mid \text{rank}(\varphi_4 \otimes k(y)) = 0 \} \]

\[ = \det \begin{pmatrix} C_1 & C_2 & C_3 & C_4 \\ 1 & C_1 & C_2 & C_3 \\ 0 & 1 & C_2 & C_3 \\ 0 & 0 & 0 & C_1 \end{pmatrix} = C_1^4 - 3C_1^2C_2 + 2C_1C_3 + C_2^2 - C_4 \]

The Giambelli-Thom-Porteous formula

\[ \text{where } C_i(\mathcal{O}(3) - \mathcal{E}) \text{ is the } i\text{-th} \]

Chern class.

\[ C_i(\mathcal{O}(3) - \mathcal{E}) = C_i(\mathcal{O}(3)) \cdot S_i(\mathcal{E}) \]
Theorem $A_4$ (Quadruple point formula)

$X^3 \subset \mathbb{P}^5$

- $q_4(X) = \frac{1}{64}$ of $X$ passing through a general point in $\mathbb{P}^5$
- $q_4(X) = \frac{1}{24}(d^4 - 6d^3 + 11d^2 - 54d + 72) + \frac{\pi^2}{2}$
- $+ 2\chi(\omega_{X_H})^{rd} + 6\chi(\omega_X) - 9\chi(\omega_{X_{,H}})$
- $- \frac{\pi}{2}(d^2 - 5d + 7)$

Example:

<table>
<thead>
<tr>
<th>Example</th>
<th>$q_4(X)$</th>
<th>$\deg(X)$</th>
<th>$\pi$</th>
<th>$\chi(\omega_X)$</th>
<th>$\chi(\omega_{X,H})$</th>
<th>$h^0(\omega_X^{(3)})$</th>
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</thead>
<tbody>
<tr>
<td>Castelnuovo 3-fold</td>
<td>0</td>
<td>5</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>$\neq 0$</td>
</tr>
<tr>
<td>Bondiga 3-fold</td>
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<td>3</td>
<td>1</td>
<td>1</td>
<td>$\neq 0$</td>
</tr>
<tr>
<td>Palatini scroll</td>
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<td>7</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>K3 scroll</td>
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<td>9</td>
<td>8</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

- Question (continued)

when $q_4(X) = 0$?

(Honestly speaking, I don't know, but I'd like to remark something about this.)
Theorem (Giambelli-Thom-Porteous)

\( E, F \): vector bundles of rank \( m, n \) on \( X \)

\( E \otimes F \) is generated by sections

\( D_k^r(\phi) = \{ x \in X | \text{rank } \phi(x) \leq k \} \) has the

(expected) codimension for \( \phi \in \text{Hom}(E,F) \)

Then, the class of \( D_k^r(\phi) \) in the Chow ring of \( X \) is given by the formula

\[
[D_k^r(\phi)] = \det \begin{bmatrix}
C_{n-k} & C_{n-k+1} & \cdots & C_{n+m-2k-1} \\
C_{n-k-1} & C_{n-k} & \cdots & C_{n+m-2k-2} \\
\vdots & \vdots & \ddots & \vdots \\
C_{n-m+1} & \cdots & C_{n-k-1} & C_{n-k}
\end{bmatrix}
\]

For example, \( k = 0, m = 4, n = 1 \)

\( D_0(\phi) = \begin{bmatrix}
C_1 & C_2 & C_3 & C_4 \\
1 & C_1 & C_2 & C_3 \\
0 & C_0 = 1, C_1 & C_2 \\
0 & 0 & 0 & 1, C_1
\end{bmatrix} \), \( C = C(E-F) \),

\( \det = C(E)/C(F) \)
(proof of Thm A₁)

We have only to compute $C_i(\sigma_{\mathbb{P}^4} - \varepsilon)$ & plug these into the Giambelli-Thom-Porteous formula.

- $C_i(\pi^* \Theta_X)$ can be computed using the Grothendieck-Riemann-Roch thm, i.e.
  \[ \text{ch}(\pi^* \Theta_X) \cdot \text{td}(\mathbb{P}^4) = \pi^* (\text{ch}(\Theta_X) \cdot \text{td}(\mathbb{P}^4)) \]

- $K_X \cdot H^2 = 2\pi - 2 - 2d$ (Adjunction formula)

- $\frac{1}{24} C_1 C_2 = \chi(\Theta_X)$ (Noether formula)

- $C_2(X) = (15 - d) H^2 + 6HK_X + K_X^2$ (see example 4.1.3, [Ha, 433p]).

- $K_X^2 \cdot H = \frac{1}{2} (d^3 + d) - 9(\pi - 1) + 6 \chi(\Theta_5 = \Omega^1_X)$ (Double-point formula)
Theorem A₂
\[ X^n \subseteq \mathbb{P}^{n+2}, \quad m \leq n \]
\[ \Phi \text{ ACM (not necessary smooth)} \]
\[ S_{m+1}(X) \neq \mathbb{P}^{n+2} \]
\[ \iff \exists \text{ a hypersurface } F \text{ of degree } m \text{ containing } X. \]

Theorem A₃
\[ X^3 \subseteq \mathbb{P}^5 \]
smooth not necessary subcanonical

1. \[ S_3(X) = \mathbb{P}^5 \iff X \subseteq \text{ a cubic hypersurface}. \]

2. If \[ H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes H^1(\mathcal{O}_X) \rightarrow H^1(\mathcal{O}_X(3)) \]
\[ \& H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes H^2(\mathcal{O}_X) \rightarrow H^2(\mathcal{O}_X(3)) \]
are injective, \[ h^0(\mathcal{I}_X(3)) = 0 \]
then \[ S_4(X) = \mathbb{P}^5 \]
Remarks

1. We do not know any nontrivial example such that \( h^0(g_x^{(3)}) = 0 \), \( S_4(x) = \mathbb{P}^5 \).
   So, classification of such examples is still open.

2. (F. Zak)
   How to generalize A. Aure's work to higher codimension case?

   "What can be said about geometric & cohomological properties of a (smooth) proj. variety \( X^m \subseteq \mathbb{P}^N \) r.t.
   
   \[ S_{m+1}^-(x) = \mathbb{P}^N , h^0(g_x^{(m)}) = 0 \]
   for \( m \leq \frac{n}{e-1} , e = N-n \) ?"