Smooth varieties
dominated by abelian varieties

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(Joint work with Ngaiming Mok)
Theme: Finite surjective morphisms between smooth projective varieties are rare.

For any smooth projective variety $X$ of dimension $n$, we have projections

$$f : X \to \mathbb{P}_n$$

It is expected that for 'general' $X$, these are essentially the only examples (other than automorphisms). Very few results of this type are known.

How to show that the image is $\mathbb{P}_n$?

Mori's characterization of $\mathbb{P}_n$: A smooth uniruled projective variety $X$ is $\mathbb{P}_n$ if all rational curves through some point $x \in X$ have ample normal bundles.
**Theorem** (1983, Lazarsfeld) (conjectured by Remmert-Van de Ven) Let $f : \mathbb{P}_n \to X$ be a finite morphism onto a smooth projective variety. Then $X$ is $\mathbb{P}_n$.

$\mathbb{P}_n \xrightarrow{\text{Ram}(f)} \mathbb{C} \xrightarrow{f} X \xrightarrow{\text{Branch}(f)} \mathbb{C}$

$N_C$ is a quotient of $N_C$, which is ample.

**Theorem** (1999, Mok-H.) (conj. by Lazarsfeld) Let $G$ be a complex simple Lie group, $P$ be a maximal parabolic subgroup, and $f : G/P \to X$ be a finite morphism onto a smooth projective variety. If $\text{Ram}(f) \neq \emptyset$, $X$ is $\mathbb{P}_n$. 
Question Let $f : A \to X$ be a finite morphism from an abelian variety onto a smooth projective variety with $\text{Ram}(f) \neq \emptyset$. What is the structure of $X$?

Remark: There are examples with $\text{Ram}(f) = \emptyset$ where $X$ is not an abelian variety (e.g. Igusa).

Theorem (1990, Debarre) (conjectured by Paranjape-Srinivas) Let $A$ be a simple abelian variety and $f : A \to X$ be a finite morphism onto a smooth projective variety. If $\text{Ram}(f) \neq \emptyset$, $X$ is $\mathbb{P}_n$.

Definition For a curve $C$ in a smooth variety $X$, let $\text{Ann}(C)$ be the subspace of $H^0(C, T^*(X))$ annihilating tangent vectors to $C$.

Definition For an irreducible curve $C \subset A$, its toroidal hull $[C]$ is the smallest abelian subvariety whose translate contains $C$.
Lemma (Zak) If $C \subset A = \mathbb{C}^n / \Lambda$ is tangent to $C^m \subset \mathbb{C}^n$, then $\dim[C] \leq m$.

Corollary For any $C \subset A$,

$$\dim[C] \leq n - \dim \text{Ann}(C).$$

In particular, if $A$ is a simple abelian variety, $\text{Ann}(C) = 0$.

Proof of Debarre's Theorem

Use the same argument as Lazarsfeld's with

Lemma A curve $C$ in a simple abelian variety has ample normal bundle.

Gieseker's criterion for ampleness: If a vector bundle $V$ on a curve is not ample, there exists a rank-1 quotient $V \to L$ with $L \leq 0$.

If $N_C$ is not ample, we have a quotient line bundle $N_C \to L$ with $L \leq 0$. $L$ is a quotient of $T(A)$, which is generated by sections. Thus $L$ is trivial. Then the quotient $N_C \to \mathcal{O}_C$ induces an injection $\mathcal{O}_C \to N_C^*$, which corresponds to a nontrivial element of $\text{Ann}(C)$. 
For an abelian variety $A$ and an abelian subvariety $A'$, let $A' \subset T(A)$ be the distribution defined by the translates of $A'$.

**Main Theorem** Let $A$ be an abelian variety and $f : A \to X$ be a finite morphism onto a smooth projective variety with $\text{Ram}(f) \neq \emptyset$. Then we have an abelian subvariety $A' \subset A$ of dimension $k > 0$, a smooth projective variety $X'$ and a morphism $g : X \to X'$ so that

$$
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow & & \downarrow g \\
A' & \xrightarrow{f'} & X'
\end{array}
$$

where $g$ is a $\mathbb{P}_k$-fibration whose fibers are images of leaves of $A'$ and $f'$ is a finite surjective morphism.

Thus $X$ has the structure of a tower of projective bundles over a smooth projective variety which admits an etale covering by an abelian variety.
Miyaoka-Mori's criterion for uniruledness: Suppose for a generic point $x$ of a smooth projective variety $X$, there exists a curve $C$ through $x$ satisfying $K_X \cdot C < 0$. Then $X$ is uniruled.

In our setting, $K_A = f^*K_X + \text{Ram}(f)$ implies $f^*K_X = -\text{Ram}(f)$. Using the images of curves on $A$ intersecting $\text{Ram}(f)$, we see that $X$ is uniruled.
Minimal rational curves

A **minimal rational curve** on a smooth projective variety $X$ is a rational curve $C$ such that (i) deformations of $C$ cover $X$ (ii) $C$ has minimal degree (w.r.t. a fixed polarization) among rational curves satisfying (i).

**Bend-and-break:** A generic minimal rational curve $C$ is an immersed $\mathbb{P}_1$ in $X$ whose normal bundle is of the type $\mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}$. $C$ has $p$-dimensional deformations fixing a point, but does not have deformations fixing two points (or one point with a tangent vector). Note $\dim \text{Ann}(C) = n - 1 - p$.

A **minimal rational component** is a maximal irreducible family $\mathcal{K}$ of minimal rational curves on $X$. For a given $x \in X$, let $\mathcal{K}_x$ be the subfamily of members passing through $x \in X$. $\mathcal{K}_x$ may be reducible. $\dim(\mathcal{K}_x) = p$ for generic $x$. 
Varieties of minimal rational tangents

The tangent map $\tau_x : K_x \rightarrow PT_x(X)$ is defined by

$\tau_x$ is generically finite over its image for generic $x$. Let $C_x = Im(\tau_x)$ be the closure of the set of the tangent vectors at $x$ of generic members of $K_x$, called the **variety of minimal rational tangents** at $x$ associated to $K$. Let $C \subset PT(X)$ be the closure of the union of $C_x$ for all generic $x \in X$. $C$ is irreducible because $K$ is irreducible.

Let $W_x \subset T_x(X)$ be the linear span of $C_x$. The collection of $W_x$ for generic $x \in X$ defines a meromorphic distribution $W$ on $X$, called the **minimal rational system**.
Structure of $\mathcal{C}$ for $X$ in Main Theorem

Choose a minimal rational component $\mathcal{K}$ for our $X$ and let

$$\mathcal{K}_x = \mathcal{K}_x^1 \cup \cdots \cup \mathcal{K}_x^m$$
$$\mathcal{C}_x = \mathcal{C}_x^1 \cup \cdots \cup \mathcal{C}_x^m$$

be the irreducible components at generic $x \in X$. Let $C^i$ be a generic member of $\mathcal{K}_x^i$. For $s \in A$ with $x = f(s)$, let $\tilde{C}^i$ be the irreducible component of $f^{-1}(C^i)$ through $s$, and $A^i := [\tilde{C}^i]$. We have $m$ distributions $A^1, \ldots, A^m$ on $A$. 

![Diagram of Structure of C for X in Main Theorem](image-url)
Lemma Let \( \{ C_\sigma, \sigma \in \Sigma \} \) be an irreducible family of curves in an abelian variety sharing a common point \( o \). Then for any two general \( \sigma, \sigma' \in \Sigma, [C_\sigma] = [C_{\sigma'}] \). In particular, \( \cup_{\sigma \in \Sigma} C_\sigma \subset [C_{\sigma'}] \) for a generic \( \sigma' \).

For each member of \( K_x^i \), the component of its inverse image through \( s \) is contained in the leaf of \( A_s^i \), implying

\[
df_s^{-1}(C_x^i) \subset P.A_s^i.
\]

From \( f^* \text{Ann}(C_x^i) \subset \text{Ann}(C_x^i) \), we get

\[
\dim(A_s^i) \leq n - \dim(\text{Ann}(C_x^i)) \\
\leq n - \dim(\text{Ann}(C_x^i)) \\
= \dim(C_x^i) + 1,
\]

concluding

\[
df_s^{-1}(C_x^i) = P.A_s^i.
\]

So \( C_x \) consists of \( m \) linear subspaces. \( C \) defines a multi-valued meromorphic foliation on \( X \) whose leaves are the images under \( f \) of translates of an abelian subvariety in \( S \) of dimension \( p + 1 \).
In fact, from the irreducibility of $C$, we see that $f_*(A^i) = C$ for any $i$. 
Structure of a generic leaf of $C$

Let $Z$ be the image of a generic translate of $A^1$. For a generic point $x \in Z$, members of a component of $\mathcal{K}_x$ cover $Z$. Let $C \subset Z$ be a generic member.

1. The normalization of $Z$ is smooth near $C$.

2. The normalization map of $Z$ cannot be ramified at $C$.

Thus $Z$ is an immersed submanifold of $X$ in a neighborhood of $C$. The normal bundle of $C$ in $Z$ is ample of rank $p$. The normal bundle of $Z$ in $X$ is trivial when restricted to $C$. 
Key Lemma On $X$, at least outside a set of codimension $\geq 2$, $C_x$ consists of $m$ distinct linear subspaces.

Proof. Suppose $C_x$ consists of less than $m$ linear subspaces at a generic point $x$ of $\text{Branch}(f)$. This means that there are two leaves $Z_1, Z_2$ of $C$ through $x$ which have the same tangent space at $x$. Assume that they are not contained in $\text{Ram}(f)$. Let $A_1, A_2$ be their inverse image through $o \in \text{Ram}(f)$ over $x$. Since $A_1, A_2$ have different tangent spaces at $o$, but their images have the same tangent spaces at $x$, we have $\text{Ker}(df_o) \subseteq T_0(A_1) \cap T_0(A_2)$.
Suppose $Z_1$ and $Z_2$ are not tangent to $\text{Branch}(f)$. Since $f|_{\text{Ram}(f)}$ is unramified at $o$,

\[
T_0(A_1) \cap \text{Ram}(f) = df^{-1}(T_x(Z_1) \cap \text{Branch}(f)) = df^{-1}(T_x(Z_2) \cap \text{Branch}(f)) = T_0(A_2) \cap \text{Ram}(f)
\]

which gives $T_0(A_1) = T_0(A_2)$, a contradiction.

Thus $Z_1$ and $Z_2$ are tangent to $\text{Branch}(f)$. From the genericity of $x$, the same holds for small deformation of $Z_1$. Thus the normal bundle of $Z_1$ near a generic $C \subset Z_1$ cannot be trivial, a contradiction.

If $Z_1$ or $Z_2$ lies on $\text{Ram}(f)$, it is easy to get a contradiction from the unramifiedness of $f|_{\text{Ram}(f)}$ at $o$. 

\[\text{Branch}(f)\]
Proof of Main Theorem when $\mathcal{W} = T(X)$

Assume there exists a minimal rational component $\mathcal{K}$ for our $X$ so that $\mathcal{W} = T(X)$. It is not difficult to show that $X$ is rationally connected, so simply connected. Since $\mathcal{C} \to X$ has $m$ distinct fiber components outside a set of codimension $\geq 2$. $\mathcal{C}$ consists of $m$ distinct components. By the irreducibility of $\mathcal{C}$, $m = 1$ and $\mathcal{C}_x = PT_x(X)$.

Refined form of Mori's characterization of $P_n$: If the tangent map $\tau_x : \mathcal{K}_x \to PT_x(X)$ is dominant and birational for a generic $x \in X$, $X = P_n$.

Thus if $X \neq P_n$, we may assume that there exists a ramification divisor $\mathcal{H}_x \subset \mathcal{K}_x$ so that $\tau_x(\mathcal{H}_x)$ is a hypersurface of degree $\geq 2$ in $PT_x(X)$. Applying the previous argument to $\mathcal{H}_x$, we see that $\tau_x(\mathcal{H}_x)$ is a hyperplane which is the image of the tangent to an abelian hyperplane in $S$, a contradiction. Thus $X = P_n$. 
Proof of Main Theorem for general $X$

Choose a minimal rational component $K$ on $X$ and let $W$ be the corresponding minimal rational system. Then $W$ is the image of a distribution on $A$ given by the linear span of the abelian subvarieties $A^1, \ldots, A^m$. Thus $W$ is integrable whose leaves are images of translates of the abelian subvariety $A'$ generated by $A^1, \ldots, A^m$.

Let $X'$ be the (normalized) moduli of the leaves of $W$ and $\beta : U \to X', \gamma : U \to X$ be the universal family. $\gamma$ is finite and birational, so an isomorphism, inducing $g : X \to X'$.

A generic fiber $X_0$ of $g$ is the image of the abelian variety $A'$. Since $X_0$ is the leaf of $W$, $W|_{X_0} = PT(X_0)$. But the members of $K$ lying on $X_0$ gives a minimal rational component for $X_0$. It follows that $X_0 = P_k$ for some $k$ from Main Theorem for the case $W = PT(X)$. 
Every fiber of $g$ is irreducible because it is the image of a translate of $A'$.

**Lemma** Let $X$ be a smooth projective variety and $\psi : X \to X'$ be a surjective morphism onto a normal variety. Assume that the underlying reduced variety of each fiber of $\psi$ is irreducible of dimension $k$. If a generic fiber is isomorphic to $\mathbb{P}_k$, then each fiber is isomorphic to $\mathbb{P}_k$ and $X'$ is smooth.

Thus $g : X \to X'$ is a $\mathbb{P}_k$-bundle, whose fibers are images of translates of $A'$ under $f$. So we have the naturally induced morphism $A/A' \to X'$ satisfying Main Theorem.