Wild $p$-cyclic actions
on surfaces with $p_g = q = 0$

$k = \overline{k}$, $\text{char}(k) = p > 0$.

$\mathcal{X}$: Smooth projective surface with $p_g = q = 0$, not nec. minimal.

$\text{gcd} \mathcal{X}$ an automorphism of order $p = \text{char}(k)$

"Wild action"

Thm
(1) $\mathcal{X}^g$ connected, i.e. a point or a connected curve
(2) $\mathcal{Y} = \mathcal{X}/\langle g \rangle$ at most rational sing.
   Its minimal resolution $\tilde{\mathcal{Y}}$ has $p_g = q = 0$. 
This is an application of

“Wild p-cyclic actions on K3 surfaces”
by I. Dolgachev and Keum

Rem. $p + |G| = q^n \Rightarrow X^G = \emptyset$ or
contains at least 2 pts.

§1. G-equivariant cohomology sheaves
§2. Artin-Schreier covering
§3. Proof
§4. Application to rational elliptic surfaces.
1. $G$-equivariant cohomology

$G$ a finite gp acting on a top. space $X$.

$\pi : X \to Y = X/G$ the quotient map.

$\mathcal{S}(X,G) = \text{cat. of abelian } G\text{-sheaves on }X$

$A = \text{cat. of abelian groups}$

The functor

$\mathcal{S}(X,G) \to A, \mathcal{F} \mapsto \Gamma'(X,\mathcal{F})^G$

can be decomposed in two ways

$\mathcal{S}(X,G) \to \mathcal{S}(Y) \to A, \mathcal{F} \mapsto \pi'_*\mathcal{F} \to \Gamma(Y,\pi^*_Y\mathcal{F})$

$\mathcal{S}(X,G) \to A \to A, \mathcal{F} \mapsto \Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{F})^G$

where $\pi'_*\mathcal{F} : U \mapsto \Gamma(\pi^*(U),\mathcal{F})^G, U \subset Y$,

(Grothendieck)

$E_2^{p,q}(\mathcal{F}) = H^p(Y, H^q(G,\mathcal{F})) \Rightarrow H^n$

$E_2^{p,q}(\mathcal{F}) = H^p(G, H^q(X,\mathcal{F})) \Rightarrow H^n$

where $H^q(G,\mathcal{F})(U) = H^q(G,\mathcal{F}(\pi^*(U))), U \subset Y$.

Apply this to:

$X$ irreducible algebraic variety / $k$, with a finite gp $G$ of automorphisms. $\mathcal{F} = \mathcal{O}_X$. 
Lemma 1. $A^i(G, O_x)$ is a torsion $O_Y$-module.

It is zero over the quotient of the open of $X$ where $G$ acts freely.

Assume $G$ cyclic of order $n$, generated by $g$.

For a $G$-module $M$,

$H^0(G, M) = \text{Ker}(g-1) = M^G$

$H^{\text{odd}}(G, M) = \text{Ker} T / \text{Im} (g-1)$

$H^{\text{even}}(G, M) = \text{Ker}(g-1) / \text{Im} T$

where $T = 1 + g + g^2 + \cdots + g^{n-1}$, $g-1 \in \mathbb{Z}[G]$.

Globalizing this fact,

$T : \pi_* O_x \to O_Y$, $g-1 : \pi_* O_x \to \pi_* O_x$.

$H^0(G, O_x) = O_Y$

$H^1(G, O_x) = \text{Ker} T / \text{Im} (g-1)$

$H^2(G, O_x) = \text{Ker}(g-1) / \text{Im} T = O_Y / \text{Im} T$
Lemma 2. If \( G \) cyclic, \( X, Y \) both Cohen-Mac. Then
\[
0 \to \omega_Y \to (\pi_X^* \omega_X)^G \to \text{Ext}_{O_Y}^1 (H^2(G, O_X), \omega_Y) \cong
\]

pf. \( \phi : \omega_Y \to (\pi_X^* \omega_X)^G \) injective (X separable) \( \pi_X^* \omega_X = \text{Hom}_{O_Y} (\pi_X^* O_X, \omega_Y) \). 
\( \phi \) is dual to \( T : \pi_X^* O_X \to O_Y \).
\[
0 \to \text{Hom}(O_Y, \omega_Y) \to \text{Hom}(\text{Im}T, \omega_Y) \to \text{Ext}^1(O_Y/\text{Im}T, \omega_Y) \to
\]
\[
\cong
\text{Hom}(\pi_X^* O_X/\text{Im}(\phi), \omega_Y) \cong (\pi_X^* \omega_X)^G
\]
\[
\text{Hom}(\text{Ker}T/\text{Im}(\phi), \omega_Y) = 0
\]
\[
H^1(G, O_X)
\]
Corollary \( \text{codim}X \geq 2 \) or \( n \in k^* \Rightarrow \omega_Y = (\pi_X^* \omega_X)^G \)

This section holds for arbitrary dimension.
2. Artin – Schreier coverings of surfaces

Well known: Any cyclic extension of deg $p$ of a field $K$ in char $p \equiv K[t]/(t^p - t - a)$.

Globalizing this fact, (In fact, modifying Takeda)

- There is a canonical filtration
  \[ O_Y = F_0 \subset F_1 \subset \cdots \subset F_{p-1} = \pi_* O_X, \]
  \[ L_i = F_i/F_{i-1} \text{ are ideal sheaves in } O_Y, \]
  \[ L_i \subset L_{i+1} \subset L_1, \text{ Im } T = L_{p-1} \]
  \[ F_i, L_i: \text{ locally free over nonsingular locus} \]
- Outside a finite set $S$ in $Y$, $L_i$ are locally free, $L_i = L_i^e$
  There exists an open affine cover $\{U_\alpha\}$ of $Y - S$.
  \[ \pi^{-1}(U_\alpha) \equiv \text{Spec } O_Y(U_\alpha)[t_\alpha]/(t_\alpha^p - a_\alpha t_\alpha - b_\alpha), \]
  \[ a_\alpha = s^{p-1}, s \in L_i^{-1} \]
  The group $G$ acts by $t_\alpha \mapsto t_\alpha + s\alpha$, $s\alpha = s|_{U_\alpha}$

# Outside a finite set, $X$ lives in the total space of $(L_i^{-1})^{p-1}$. Define $L = L^{**}$.
There exists a positive Weil divisor $B$ s.t. $L^{-1} \equiv O_Y(B)$
Call $B$ the branch divisor of $\pi: X \to Y$. 
Corollary. Outside a finite set of points in $Y$,
\[ H^1(G, O_Y) = O_Y / \text{Im}(T) \cong O_Y(p-1)B \]

pf. Take the open subset $Y'$ where $L = L_1$.
On $Y'$, \( \text{Im}(T) = L_{p-1} = L_1^{p-1} = O_Y(-1(p-1)B) \)

Corollary. \( W_X = \pi^*(W_Y \otimes O_Y(p-1)B) \)

Remark. This is different from the one in char 0.
pf. Both reflexive, so enough to verify this formula over the complement of a finite set.
Then this follows from \#.

This section holds true for higher dimensions
if we replace "finite" by "codimension 2".

Example \[ g : \mathbb{A}^2 \rightarrow \mathbb{A}^2, \ u \rightarrow u, \ v \rightarrow v+u \]
char = 3. \[ W = v \cdot g(v) \cdot g^*(v) = v^3 - u^2v \]
u, w, algebraically independent, \( \in k[u, v] \)
\[ \pi^*(du \wedge dw) = -u^2 du \wedge dv \]
3. Proof of Theorem.

Lemma. \( H^2(Y, O_Y) = 0 \)

pf. \( H^0(Y, (\pi_x W_x)^G) = H^0(X, W_x)^G = 0 \)
\( H^0(Y, W_Y) = H^2(Y, O_Y) = 0 \).

Lemma. \( H^1 = H^2 = k \)

proof.

Standard five term sequence

\[ 0 \rightarrow E_2^{1,0} \rightarrow H^1 \rightarrow E_2^{0,1} \rightarrow E_2^{2,0} \rightarrow H^2 \]

Apply this to \( E_2^{p,q} = H^p(G, H^q(X, O_X)) \)

to get

\[ E_2^{0,1} = H^0(G, H^1(X, O_X)) = 0 \]
\[ E_2^{1,0} = H^1(G, k) = \text{Hom}(G, k) = k \]
\[ E_2^{2,0} = H^2(G, k) = \text{Ker}(\partial_1) / \text{Im} \partial_1 = k \]

Can extend \( \text{[Cartan - Eilenberg]} \)

\[ \rightarrow E_2^{2,0} \rightarrow H^2 \rightarrow E_2^{0,2} = 0 \]
Recall $H^2(G, O_x) = O_Y / \text{Im } T$

$Z := \text{the closed subscheme of } Y$

defined by $\text{Im } T$

$\Rightarrow H^2(G, O_x) = O_Z$

**Lemma.** \( Z_{\text{red}} = \pi(X^g) \)

**proof.**

$H^2 = 0$ over the quotient of the open

of $X$ where $G$ acts freely

$\Rightarrow Z_{\text{red}} \subseteq \pi(X^g)$

Let \( x \in X^g \).

$g$ acts on $O_x, x$, $g(m_x) \leq m_x$

$T(O_x, x) \leq m_x \cap O_Y, \pi(x) = m_y, \pi(x)$

$\therefore T \text{ kills constant functions}$

$\Rightarrow x \in Z_{\text{red}}$
Case 1. Suppose $X^3$ finite.

- $H^0(Y, H^i(G, O_X)) \cong \bigoplus_{x \in X^3} H^i(G, O_{x, X})$
- $\forall x \in X^3$, isolated, $H^1(G, O_{x, X}) \neq 0$.
- $H^0(Y, H^1(G, O_X)) \cong k \Rightarrow |X^3| = 1$
- [B. Peskin] $H^1(G, O_{x, X}) = k \Rightarrow$ rational sing.

- Let $\sigma: \tilde{Y} \to Y$ min resolution.

\[ 0 \to H^1(Y, O_Y) \to H^1(\tilde{Y}, O_{\tilde{Y}}) \to H^0(Y, R^1\sigma_* O_{\tilde{Y}}) \]
\[ \to H^2(Y, O_Y) \to H^2(\tilde{Y}, O_{\tilde{Y}}) \to H^0(Y, R^2\sigma_* O_{\tilde{Y}}) \]
Case 2. Suppose $X^3$ contains a 1-dimensional part.

- $H^0(Y, H^2(G, O_X)) \cong k$
  (use the spectral sequence.)

- $Z_{\text{red}}$ connected ($\therefore H^2(G, O_X) \cong O_Z$)

\[ \pi(X^3) \]

\[ \begin{array}{ccc}
W & \xrightarrow{f} & \tilde{X} \\
\downarrow & & \downarrow \pi \\
\tilde{Y} & \xrightarrow{\delta} & Y
\end{array} \]

$X$ the normalization of $\tilde{Y}$ in $\mathbb{A}(X)$

$W \to \tilde{X}$ resolution

$H^1(W, O_W) \cong H^1(X, O_X) \cong H^1(\tilde{X}, O_{\tilde{X}}) = 0$

$R^1f_*O_W = 0$
0 → O₀ → πₓ(Oₓ) → Im(g⁻¹) → 0

9⁻¹ ⊗ πₓOₓ : O₀ -module hom.

⇒ \( H^1(\tilde{Y}, \pi_* O_X) = H^1(\tilde{X}, O_X) = 0 \)

(\( \pi \): finite)

- \( H^1(\tilde{Y}, O_{\tilde{Y}}) = 0 \) (\( \therefore H^0(\tilde{Y}, \text{Im}(g⁻¹)) = 0 \))
  Im(g⁻¹) has filtration with quotients \( L_i \subset L_i \),
  but \( L_1 = O_{\tilde{Y}} (-B) \)
  away from a codim 2 set.

- The Leray spectral sequence for \( \sigma : \tilde{Y} \to Y \)
  completes the proof.

$\phi: S \to \mathbb{P}^1$ rational elliptic surface

with a $p$-torsion section

Then

(1) If $\phi$ has a fibre of multiplicative type in
    then $p$ divides $n$.

(2) $\phi$ has exactly one fibre of additive type

    $\mathbb{I}$, $\mathbb{II}$, $\mathbb{IV}$, $\mathbb{I}^*$, $\mathbb{IV}^*$, $\mathbb{III}^*$, $\mathbb{II}^*$

Cor. Assume $p = \text{char } k > 5$. Then

$\phi$ has no $p$-torsion section.

Rem. Oguiso & Shioda classified

all pair $(T, E(K))$, $T$ trivial lattice or the types of Sing

$E(K)$ Mordell-Weil group

$\exists 74$ cases, some of them cannot occur if $p$ divides the order of $E(K)_{\text{tor}}$. 