IR-Moduli-Space in Analytic Category

Thm. (WAVRICK) $\mathcal{X} \rightarrow \mathcal{M}$

Kuranishi family and $H^0(\Theta_{X_t})$ indep. of $t \Rightarrow$

$\mathcal{B}/\text{Aut}(X_0)$ is the Local Moduli-Space $M_{x_0}$

$\sigma : X_0 \rightarrow \bar{X}_0$ anti hol.

$\mathcal{B}(IR) = \mathcal{B} \cap \{ J | \sigma_2 J \sigma_2 = -J \}$

$M_{x_0} (IR) = \mathcal{B}(IR) / \text{Aut}\sigma = \frac{\mathcal{B}}{\text{Aut}\sigma (IR)}$

$\text{Aut}\sigma = \{ \phi | \phi^* \sigma = \sigma \}$

$\phi \sigma \phi^{-1}$
What is a real variety?

Definition 0.1. A smooth real variety \((X, \sigma)\) is a pair consisting of the data of a smooth complex variety \(X\) of complex dimension \(n\) and of an involution \(\sigma : X \to X\) which is antiholomorphic.

I.e., let \(M\) be the differentiable manifold underlying \(X\) and let \(J\) be the complex structure of \(X\): then the complex structure \(-J\) determines a complex manifold which is called the conjugate of \(X\) and denoted by \(\bar{X}\), and \(\sigma\) is said to be antiholomorphic if it provides an isomorphism between the complex manifolds \(X\) and \(\bar{X}\).
What are the main problems?

(we may restrict ourselves to the case where $X$ is compact)

- Describe the isomorphism classes of such pairs $(X, \sigma)$.
- Or, at least describe the possible topological or differentiable types of the pair $(X, \sigma)$.
- At least describe the possible topological types for the real part $X' := X(\mathbb{R}) = Fix(\sigma)$.

Remark 0.2. Recall that Hilbert's 16-th problem is a special case of the last question but for the more general case of a real pair $(Z \subset X, \sigma)$.

The Enriques classification of real algebraic surfaces, not yet achieved in the strongest form
above, is however achieved for Kodaira dimension 0 (thanks to Comessatti (ca. 1911), Silhol, Nikulin, Kharlamov and Degtyarev, −).

These are our main results:

**Theorem 0.3.** Let \((S, \sigma)\) be a real hyperelliptic surface. Then the differentiable type of the pair \((S, \sigma)\) is completely determined by the orbifold fundamental group exact sequence.

**Theorem 0.4.** Fix the topological type of \((S, \sigma)\) corresponding to a real hyperelliptic surface. Then the moduli space of the real surfaces \((S', \sigma')\) with the given topological type is irreducible (and connected).
Theorem 0.5. Real hyperelliptic surfaces fall into exactly 78 topological types. In particular, the real part $S(\mathbb{R})$ of a real hyperelliptic surface is either

- a disjoint union of $c$ tori, where $0 \leq c \leq 4$
- a disjoint union of $b$ Klein bottles, where $1 \leq b \leq 4$.
- the disjoint union of a torus and of a Klein bottle
- the disjoint union of a torus and of two Klein bottles.

Up to now, hyperelliptic surfaces were attacked using the two elliptic pencils on them: our method consists in

1) Rerunning the classification theorem with special attention to the real involution.
2) Finding out the primary role of the orbifold fundamental group, which we shall now introduce.

For $X$ real smooth, we have a double covering $\pi : X \to Y = X/\langle \sigma \rangle$ ($Y$ is called the Klein variety of $(X, \sigma)$), ramified on the real part of $X$, $X' := X(\mathbb{R}) = \text{Fix}(\sigma)$. Set $Y' = \pi(X')$.

Either $X' = \emptyset$ or $X'$ is a real submanifold of dimension $n$, hence $\pi_1(X - X') \to \pi_1(X)$ is onto for $n \geq 2$ and an isomorphism for $n \geq 3$.

If $\dim_{\mathbb{C}} X = n \geq 3$, or if $Y'$ is empty, $\pi_1^{\text{orb}}(Y)$ is just defined as the fundamental group of $Y - Y'$, whereas, in case $n = 2$, $\pi_1^{\text{orb}}(Y')$ is defined to be the quotient of $\pi_1(Y - Y')$ by the smallest normal subgroup containing the squares $\gamma^2$ of the local loops $\gamma$ around the components of $Y'$. 
If finally \( n = 1 \), and \( X(\mathbb{R}) = \text{Fix}(\sigma) \neq \emptyset \), take a fixed point \( x_0 \) and since \( \sigma \) acts on \( \pi_1(X, x_0) \), define \( \pi_1^{\text{orb}}(Y) \) as the semidirect product of \( \pi_1(X, x_0) \) with \( \mathbb{Z}/2 \) (generated by \( \sigma \)).

We have thus in all cases an exact sequence

\[
1 \rightarrow \pi_1(X) \rightarrow \pi_1^{\text{orb}}(Y) \rightarrow \mathbb{Z}/2 \rightarrow 1.
\]

This sequence is very important when the universal cover of \( X \) is contractible:

**Question 1**: to what extent does then \( \pi \) also determine the differentiable type of \( X \), not only the homotopy type?

**SLOGAN**: analogously, from the differentiable viewpoint, real varieties should be described in terms of the orbifold fundamental group.

There are other similar instances in the Kodaira classification of real surfaces (forthcoming work).
Question 2: determine, for the real varieties whose diff. type is determined by the orbifold fundamental group, those for which the moduli spaces are irreducible and connected.

Observe: a hyperbola \( \{(x, y) \in \mathbb{R}^2 | xy = 1\} \) is irreducible but not connected.

I want to show an easy example, leading to the quotient of the above hyperbola by the involution \((x, y) \to (-x, -y)\) as moduli space, and explaining the basic philosophy underlying the two theorems concerning the orbifold fundamental group.

1. REAL ELLIPTIC CURVES

Classically, the topological type of real elliptic curves is classified according to the number
\[ \nu = 0, 1, \text{ or } 2 \] of connected components (circles) of their real part.

The orbifold fundamental group sequence is here

\[ 1 \to H_1(C, \mathbb{Z}) \equiv \mathbb{Z}^2 \to \pi_1^{orb}(C) \to \mathbb{Z}/2 \to 1. \]

and it splits iff \( C(\mathbb{R}) \neq \emptyset \) (\( \pi_1^{orb}(C) \) has a representation as a group of affine transformations of the plane).

Step 1: If there are no fixed points, the action \( s \) of \( \mathbb{Z}/2 \) on \( \mathbb{Z}^2 \) is diagonalizable.

**PROOF:**

let \( \sigma \) be represented by the affine transformation \( (x, y) \to s(x, y) + (a, b) \). Now, \( s \) is not diagonalizable if and only if \( s(x, y) = (y, x) \). The square of \( \sigma \) is the transformation \( (x, y) \to (x, y) + (a + b, a + b) \), thus \( a + b \) is an integer, and therefore the points \( (x, x - a) \) yield a fixed
$S^1$ on the elliptic curve. Q.E.D.

Step 2: moreover, then, the translation vector of the affine transformation inducing $\sigma$ can be chosen to be $1/2$ of the $+1$-eigenvector $e_1$ of $s$.

Step 3: there are thus only 3 normal forms! Moreover, $s$ is diagonalizable if and only if $\nu$ is even.

A DESCRIPTION OF THE MODULI SPACE FOR THE SPLIT CASE WITH $\nu = 1$.

$\sigma$ acts as follows: $(x, y) \rightarrow (y, x)$. We look then for the complex structures $J$ which make $\sigma$ antiholomorphic, i.e., we seek for the matrices $J$ with $J^2 = -1$ and with $Js = -sJ$.

The latter condition singles out the matrices

$$\begin{pmatrix} a & b \\ -b & -a \end{pmatrix}$$
while the first condition is equivalent to requiring that the characteristic polynomial be equal to $\lambda^2 + 1$, whence, equivalently, $b^2 - a^2 = 1$.

We get a hyperbola with two branches which are exchanged under the involution $J \rightarrow -J$, but, as we already remarked, $J$ and $-J$ yield isomorphic real elliptic curves, thus the moduli space is irreducible and connected. Q.E.D.

2. Real hyperelliptic surfaces on the scene

Definition 2.1. A complex surface $S$ is said to be hyperelliptic if $S \cong (E \times F)/G$, where $E$ and $F$ are elliptic curves and $G$ is a finite group of translations of $E$ with a faithful action on $F$ such that $F/G \cong \mathbb{P}^1$. 
Theorem 2.2. (Bagnera - de Franchis)
Every hyperelliptic surface is one of the following, where \( E, F \) are elliptic curves and \( G \) is a group of translations of \( E \) acting on \( F \) as specified:

1. \((E \times F)/G, G = \mathbb{Z}/2\) acts on \( F \) by \( x \mapsto -x \).

2. \((E \times F)/G, G = \mathbb{Z}/2 \oplus \mathbb{Z}/2\) acts on \( F \) by \( x \mapsto -x, x \mapsto x + \epsilon \), where \( \epsilon \) is a half period.

3. \((E \times F_i)/G, G = \mathbb{Z}/4\) acts on \( F_i \) by \( x \mapsto ix \).

4. \((E \times F_i)/G, G = \mathbb{Z}/4 \oplus \mathbb{Z}/2\) acts on \( F_i \) by \( x \mapsto ix, x \mapsto x + (1 + i)/2 \).

5. \((E \times F_\rho)/G, G = \mathbb{Z}/3\) acts on \( F_\rho \) by \( x \mapsto \rho x \).

6. \((E \times F_\rho)/G, G = \mathbb{Z}/3 \oplus \mathbb{Z}/3\) acts on \( F_\rho \) by \( x \mapsto \rho x, x \mapsto x + (1 - \rho)/3 \).
7. \((E \times F_p)/G, G = \mathbb{Z}/6\) acts on \(F_p\) by \(x \mapsto -px\).

**Theorem 2.3.** The complex surfaces \(S\) with \(K\) nef, \(K^2 = 0, p_g = 0\) and such that either \(S\) is algebraic with \(q = 1\), or more generally \(b_1 = 2\), are hyperelliptic surfaces if and only if \(\text{kod}(S) = 0\) (equivalently, the Albanese fibres are smooth of genus 1).

**Definition 2.4.** The extended symmetry group \(\hat{G}\) is the group generated by \(G\) and \(\sigma\).

Rerunning the classification theorem yields

**Theorem 2.5.** Let \((S, \sigma), (\hat{S}, \hat{\sigma})\) be isomorphic real hyperelliptic surfaces: then the respective extended symmetry groups \(\hat{G}\) are the same and given two Bagnara - De Franchis
realizations $S = (E \times F)/G$, $\hat{S} = (\hat{E} \times \hat{F})/\hat{G}$, there is an isomorphism $\Psi : E \times F \rightarrow \hat{E} \times \hat{F}$, of product type, commuting with the action of $\hat{G}$, and inducing the given isomorphism $\psi : S \cong \hat{S}$. Moreover, let $\tilde{\sigma} : E \times F \rightarrow E \times F$ be a lift of $\sigma$. Then the antiholomorphic map $\tilde{\sigma}$ is of product type.

Let us give the list of all the possible groups $\hat{G}$.

**Lemma 2.6.** Let us consider the extension

\[(*) \quad 0 \rightarrow G \rightarrow \hat{G} \rightarrow \mathbb{Z}/2 = \langle \sigma \rangle \rightarrow 0.\]

We have the following possibilities for the action of $\sigma$ on $G$: in what follows the subgroup $T$ of $G$ will be the subgroup acting by translations on both factors.

1. If $G = \mathbb{Z}/q$, $q = 2, 4, 3, 6$ then $\sigma$ acts as $-Id$ on $G$ and $\hat{G} = D_q$
2. If $G = \mathbb{Z}/2 \times \mathbb{Z}/2$, then either $\sigma$ acts as the identity on $G$ and $\hat{G} = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$, if (*) splits, $\hat{G} = \mathbb{Z}/4 \times \mathbb{Z}/2$ if (*) does not split, and in this latter case the square of $\sigma$ is the generator of $T$.

Or $\sigma$ acts as \[
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix},
\] the sequence splits, $\hat{G} = D_4$, the dihedral group, and again the square of a generator of $\mathbb{Z}/4\mathbb{Z}$ is the generator of $T$.

3. If $G = \mathbb{Z}/4 \times \mathbb{Z}/2$, then either $\hat{G} = T \times D_4 \cong \mathbb{Z}/2 \times D_4$, or $\hat{G}$ is isomorphic to the group $G_1 := < \sigma, g, t, | \sigma^2 = 1, g^4 = 1, t^2 = 1, t\sigma = \sigma t, tg = gt, \sigma g = g^{-1}t\sigma >$, and its action on the second elliptic curve $F$ is generated by the following transformations: $\sigma(z) = \bar{z} + 1/2$, $g(z) = iz$, \[...\]
t(z) = z + 1/2(1 + i). The group $G_1$ is classically denoted by $c_1$ (cf. Atlas of Groups).

In particular, in both cases (*) splits.

4. If $G = \mathbb{Z}/3 \times \mathbb{Z}/3$, then we may choose $G'$ so that $\sigma$ acts as $-\text{Id} \times \text{Id}$ on $G = G' \times T$ and $\hat{G} = D_3 \times \mathbb{Z}/3$.

MAIN IDEAS OF THE CLASSIFICATION THEOREM:

to distinguish the several cases, we use byproducts of the orbifold fundamental group, such as the structure of the extended Bagnara de Franchis group $\hat{G}$, parities of the involutions in $\hat{G}$, their actions on the fixed point sets of transformations in $G$, the topology of the real part, and, in some special cases, some more refined invariants such as the translation parts of all possible
lifts of a given element in $\hat{G}$ to the orbifold fundamental group (which has a faithful representation as a group of affine transformations).

**HOW TO CALCULATE THE TOPOLOGY OF THE REAL PART?**

Use the Albanese variety and what we said about the real parts of elliptic curves, that they consist of $\nu \leq 2$ circles. Since also the fibres are elliptic curves, we get up to 4 circle bundles over circles, thus the connected components are either 2-Tori or Klein bottles.

For every connected component $V$ of $\text{Fix}(\sigma) = S(R)$ the inverse image $\pi^{-1}(V)$ splits as the $G$-orbit of any of its connected components. Let $W$ be one such: then one can easily see that there is a lift $\bar{\sigma}$ of $\sigma$ such that $\bar{\sigma}$ is an involution and $W$ is in the fixed locus of $\bar{\sigma}$: moreover $\bar{\sigma}$ is unique.
Thus the connected components of $\text{Fix}(\sigma)$ correspond bijectively to the set $\mathcal{C}$ obtained as follows: consider all the lifts $\bar{\sigma}$ of $\sigma$ which are involutions and pick one representative $\bar{\sigma}_i$ for each conjugacy class.

Then we let $\mathcal{C}'$ be the set of equivalence classes of connected components of $\bigcup \text{Fix}(\bar{\sigma}_i)$, where two components $A, A'$ of $\text{Fix}(\bar{\sigma}_i)$ are equivalent if and only if there exists an element $g \in G$, such that $g(A) = A'$.

Let now $\bar{\sigma}$ be an antiholomorphic involution which is a lift of $\sigma$ and such that $\text{Fix}(\bar{\sigma}) \neq \emptyset$.

Since $\bar{\sigma}$ is of product type, $\text{Fix}(\bar{\sigma})$ is a disjoint union of $2^{a_1+a_2}$ copies of $S^1 \times S^1$, where $a_i \in \{0, 1\}$.

In fact if an antiholomorphic involution $\bar{\sigma}$ on an elliptic curve $C$ has fixed points, then $\text{Fix}(\bar{\sigma})$ is
a disjoint union of $2^a$ copies of $S^1$, where $a = 1$ if the matrix of the action of $\delta$ on $H_1(C, \mathbb{Z})$ is diagonalizable, else $a = 0$.

What said insofar describes the number of such components $V$; in order to determine their nature, observe that $V = W/H$, where $W$ is as before and $H \subset G$ is the subgroup such that $HW = W$.

Since the action of $G$ on the first curve $E$ is by translations, the action on the first $S^1$ is always orientation preserving, whence $V$ is a Klein bottle if and only if $H$ acts on the second $S^1$ by some orientation reversing map, or equivalently $H$ has some fixed point on the second $S^1$.

We have however a restriction: let $h \in H$ be a transformation having a fixed point on the second $S^1$. 
Since the direction of this $S^1$ is an eigenvector for the tangent action of $h$, it follows that the tangent action is given by multiplication by $-1$.

FINALLY, THE FOLLOWING TABLES SHOW THAT REALLY EVERYTHING CAN BE NICELY WRITTEN DOWN AS IN THE BAGNERA DE FRANCHIS LIST.

I will only comment on a strange picture, showing that the analytic classification makes the moduli space look more awkward than it really is, already in the simple case of the elliptic curves!
The Moduli Space of IR-Elliptic Curves

\[ a = \pm 1 \]

\[ \Im \tau = 1 \]

\[ \Re \tau = -\frac{1}{2} \]

\[ \sigma(z) = a \overline{z} + b \]

\[ \gamma = 1 \]

\[ \gamma = 2 \]

\[ \gamma = 0 \]

\[ b = 0 \]
If \( G = \mathbb{Z}/6 \), we take a generator \( g \) of \( G \) such that \( g(x_1, x_2) = (x_1 + \eta, -\rho x_2) \) and also here the homology of \( S(\mathbb{R}) \) is the only topological invariant needed to distinguish the four cases.

<table>
<thead>
<tr>
<th>( \mathbb{Z}/6 )</th>
<th>( x_1 = 0 ), ( \rho )</th>
<th>( z_1, \eta = \frac{1}{2} ) or ( -z_1, \eta = \frac{1}{4} )</th>
<th>( z_2, 2T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_6 )</td>
<td>( x_1 = 0 )</td>
<td>( z_1, \eta = \frac{1}{2} + \frac{1}{4} ) or ( -z_1, \eta = \frac{1}{4} + \frac{1}{4} )</td>
<td>( z_2, T )</td>
</tr>
<tr>
<td>( \mathbb{Z}/6 )</td>
<td>( x_1 = 0 ), ( \rho )</td>
<td>( z_1 + \frac{1}{2}, \eta = \frac{2}{4} ) or ( -z_1 + \frac{1}{2}, \eta = \frac{1}{4} )</td>
<td>( z_2, \emptyset )</td>
</tr>
<tr>
<td>( D_6 )</td>
<td>( x_1 = 0 )</td>
<td>( z_1, \eta = \frac{1}{2} ) or ( -z_1, \eta = \frac{1}{4} )</td>
<td>( z_2, 2K )</td>
</tr>
<tr>
<td>( \mathbb{Z}/6 )</td>
<td>( \tau_1 = 1 ), ( \rho )</td>
<td>( z_1, \eta = \frac{1}{2} ) or ( -z_1, \eta = \frac{1}{4} ) if (</td>
<td>\tau_1</td>
</tr>
<tr>
<td>( D_6 )</td>
<td>( \cup ) ( x_1 = -\frac{1}{2} )</td>
<td>( z_1, \eta = \frac{1}{2} ) or ( -z_1, \eta = \frac{1}{4} ) if ( x_1 = -\frac{1}{2} )</td>
<td>( z_2, 2K )</td>
</tr>
</tbody>
</table>

Let us now consider the case where \( G = \mathbb{Z}/2 \times \mathbb{Z}/2 \).

Here, we may choose as generators of \( G \) the generator \( t \) of \( T \) and another element \( g \), which is not canonically defined.

Let \( \eta_1, \varepsilon_1, \varepsilon_2 \) be such that \( g(x_1, x_2) = (x_1 + \eta_1, -x_2), t(x_1, x_2) = (x_1 + \varepsilon_1, x_2 + \varepsilon_2) \).

Here we have the same topological invariants 1), 2), as in the case \( G = \mathbb{Z}/2 \).

Furthermore the normal subgroup \( T \) of \( G \) is of order 2, let as usual \( t \) be a generator. We may consider all the possible liftings of \( t \) to a vector \( t' \) in the lattice \( \Omega' = \Lambda' \oplus \Gamma \).

The condition whether there exists such a \( t' \) whose two components are eigenvectors for the action of \( \sigma_1 \) on \( \Omega' \) is a topological invariant of the real hyperelliptic surface (notice that the two possible choices for \( \sigma_2 \) differ just up to multiplication by \(-1\)).
The second case is distinguished from all the others by the invariant 1)(b).
In fact the set of values of \( \nu(\sigma_2) \) is \( \{2, 0\} \) in the second case, while in the
other cases it is either \( \{2, 2\} \), or \( \{1, 1\} \) (only in the third case).

The first case is distinguished from the last one since in the first case \( \epsilon_1 \)
is an eigenvector for the action of \( \sigma_1 \) on \( \Lambda' \), while in the last case \( \epsilon_1 = \frac{1 + \eta_1}{2} \),
which is not an eigenvector for the action of \( \sigma_1 \).

\[ S(\mathbb{R}) = 2K \cup T \] the first case is distinguished from the second one since
in the first case \( \epsilon_1 \) is an eigenvector for the action of \( \sigma_1 \) on \( \Lambda' \), while in the
second case \( \epsilon_1 = \frac{1 + \eta_1}{2} \), which is not an eigenvector for the action of \( \sigma_1 \).

If \( \tilde{G} = D_4 \), the cases with \( S(\mathbb{R}) = 2T \) are distinguished by the parity of
\( \nu(\sigma_2) \), which is equal to 2 in the first case, and equal to 1 in the second case.

We finally give the table for the non split case, where \( G = \mathbb{Z}/2 \times \mathbb{Z}/2 \) is
generated by elements \( g, t \) such that \( g(x_1, x_2) = (x_1 + \eta_1, -x_2), \)
\( t(x_1, x_2) = (x_1 + \epsilon_1, x_2 + \epsilon_2) \). Recall the by now standard notation \( \tau_2 = x_j + t \eta_j \).

<table>
<thead>
<tr>
<th>( G, \tilde{G} )</th>
<th>( x_1 = 0 )</th>
<th>( x_2 = -\frac{1}{4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Z}/2 )</td>
<td>( \eta_1 = 1/4 )</td>
<td>( \epsilon_1 = 1/4 )</td>
</tr>
<tr>
<td>( \mathbb{Z}/4 \times \mathbb{Z}/2 )</td>
<td>( \eta_1 = 1/4 )</td>
<td>( \epsilon_1 = 1/4 )</td>
</tr>
</tbody>
</table>

\[ S(\mathbb{R}) \]

<table>
<thead>
<tr>
<th>( \tau_1 )</th>
<th>( \tau_2 )</th>
<th>( \sigma_1, \eta_1, \epsilon_1 )</th>
<th>( \sigma_2, \epsilon_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 )</td>
<td>( 1 )</td>
<td>( 1 )</td>
<td>( 1 )</td>
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REFERENCES


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[B-dF2] G. Bagnera, M. de Franchis "Le superficie algebriche le quali contengono una rappresentazione parametrica mediante funzioni iperellittiche di due argumen-


\[
\begin{array}{|c|c|c|c|}
\hline
G/4 & x_1 = 0, & i & x_1, \eta = \frac{1}{2} + \frac{i}{4} \\
D_4 & & or \ -x_1, \eta = \frac{1}{2} + \frac{i}{2} & Z_2, 2T \\
\hline
Z/4 & x_1 = 0, & i & x_1 + \frac{i}{2}, \eta = \frac{1}{2} + \frac{i}{4} \\
D_4 & & or \ -x_1 + \frac{i}{2}, \eta = \frac{1}{4} + \frac{i}{2} & Z_2, T \\
\hline
Z/4 & x_1 = 0, & i & x_1 + \frac{i}{2}, \eta = \frac{1}{2} + \frac{i}{4} \\
D_4 & & or \ -x_1 + \frac{i}{2}, \eta = \frac{1}{4} + \frac{i}{2} & Z_2 + \frac{1+i}{2}, T \\
\hline
Z/4 & x_1 = 0, & i & x_1, \eta = \frac{1}{2} + \frac{i}{4} \\
D_4 & & or \ -x_1, \eta = \frac{1}{4} + \frac{i}{2} & Z_2 + \frac{1+i}{2}, T \\
\hline
Z/4 & x_1 = 0, & i & x_1 + \frac{i}{2}, \eta = \frac{1}{4} + \frac{i}{4} \\
D_4 & & or \ -x_1 + \frac{i}{2}, \eta = \frac{1}{4} + \frac{i}{2} & Z_2, T \\
\hline
Z/4 & x_1 = 0, & i & x_1 + \frac{i}{2}, \eta = \frac{1}{4} + \frac{i}{4} \\
D_4 & & or \ -x_1 + \frac{i}{2}, \eta = \frac{1}{4} + \frac{i}{2} & Z_2 + \frac{1+i}{2}, 0 \\
\hline
Z/4 & x_1 = 0, & i & x_1, \eta = \frac{1}{2} + \frac{i}{4} \\
D_4 & & or \ -x_1, \eta = \frac{1}{4} + \frac{i}{2} & Z_2, 0 \\
\hline
Z/4 & x_1 = 0, & i & x_1, \eta = \frac{1}{2} + \frac{i}{4} \\
D_4 & & or \ -x_1, \eta = \frac{1}{4} + \frac{i}{2} & Z_2 + \frac{1+i}{2}, 0 \\
\hline
Z/4 & |r_1| = 1 & i & r_1 x_1, \eta = \frac{1}{4} + \frac{i}{4} \\
D_4 & \cup \ x_1 = -\frac{1}{2} & & or \ -r_1 x_1, \eta = \frac{1}{4} + \frac{i}{4} \\
\hline
& & & if |r_1| = 1, \\
& & & \bar{x}_1, \eta = \frac{1}{2} + \frac{i}{2} \\
& & & \bar{x}_1, \eta = \frac{1}{4} + \frac{i}{2} \\
& & & if x_1 = \frac{1}{2} \\
& & & \bar{x}_1, \eta = \frac{1}{4} + \frac{i}{2} \\
& & & \bar{x}_1, \eta = \frac{1}{4} + \frac{i}{2} \\
\hline
Z/4 & |r_1| = 1 & i & r_1 x_1, \eta = \frac{1}{4} + \frac{i}{4} \\
D_4 & \cup \ x_1 = -\frac{1}{2} & & or \ -r_1 x_1, \eta = \frac{1}{4} + \frac{i}{4} \\
\hline
& & & if |r_1| = 1, \\
& & & \bar{x}_1, \eta = \frac{1}{2} + \frac{i}{2} \\
& & & \bar{x}_1, \eta = \frac{1}{4} + \frac{i}{2} \\
& & & if x_1 = \frac{1}{2} \\
& & & \bar{x}_1, \eta = \frac{1}{4} + \frac{i}{2} \\
& & & \bar{x}_1, \eta = \frac{1}{4} + \frac{i}{2} \\
\hline
\end{array}
\]

In the list, if \( S(\mathbb{R}) = T \), the second and the third cases are distinguished by invariant \( 1' \)(a): in the second case the set of values of \( \nu(\sigma_1) \) equals \( \{2, 2, 2, 2\} \), in the third it equals \( \{0, 2, 0, 2\} \).

The first case is distinguished from the others by the invariant \( 1' \)(b): in the first case \( \{ \nu(\sigma_2 \circ g^n), \ for \ n = 0, 1, 2, 3 \} = \{2, 1, 2, 1\} \), while in the other cases \( \{ \nu(\sigma_2 \circ g^n), \ for \ n = 0, 1, 2, 3 \} = \{0, 1, 0, 1\} \).

If \( S(\mathbb{R}) = 0 \), the first case is distinguished from the third case by the invariant \( 1' \)(a): in the first case the set of values of \( \nu(\sigma_1) \) equals \( \{2, 0, 2, 0\} \), in the third it equals \( \{0, 0, 0, 0\} \).

The second case is distinguished from the other cases by the invariant \( 1' \)(b): in the second case the set of values of \( \nu(\sigma_2) \) equals \( \{2, 1, 2, 1\} \), in the other cases it equals \( \{0, 1, 0, 1\} \).

If \( G = \mathbb{Z}/4 \times \mathbb{Z}/2 \), we take generators \( y, t \), such that on \( F \), \( y = -x_2, \) 
\( t(x_2) = x_2 + (1 + i)/2 \).
If $G = \mathbb{Z}/3 \times \mathbb{Z}/3$, with generators $g, t$ acting on $F$ by $g(x_2) = \rho x_2$, $t(x_2) = x_2 + (1 - \rho)/3$, the homology of $S(\mathbb{R})$ is the only topological invariant needed in order to distinguish the three cases.
Elliptic Curves:

UNIFORMIZED by $C$

$(\sigma(z), \sigma'(z), 1)$

"Hyperelliptic Varieties:"

UNIFORMIZED by $C^n$

(eg: Tori)

Humbert, Picard

$n = 2$ End of the Classification

Bordin-Prize: Enriques-Solveri (1908)

Bordin-Prize: Bagnara De-Franchis (1908)