Classification of semifree monotone Hamiltonian S^1-manifolds

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Fano variety

Definition : A **Fano variety** is a complete (smooth) variety X whose anti-canonical bundle is ample. (Equivalently, $^{\exists}$Kähler form ω s.t. $-K_X = c_1(TX) = [\omega]$.)

Example :

- $\mathbb{P}^n$
- (complete intersection of) smooth hypersurfaces of deg $\leq n$
- (generalized) flag varieties

Theorem : (Kollar-Miyaoka-Mori '92) For each dimension, there are finitely many deformation types of smooth Fano varieties.

Monotone symplectic manifold

Definition : A symplectic form ω on M is **monotone** if $c_1(TM) = \lambda[\omega]$ for some $\lambda > 0$.

(Example : smooth Fano variety)

Question : Is every closed monotone symplectic manifold Fano?

- $\dim M = 2$: Yes ($M \cong \mathbb{P}^1$.)
- $\dim M = 4$: Yes (H. Ohta - K. Ono '96)
  ($M \cong \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}^2 \# k\overline{\mathbb{P}^2}$ for $k \leq 8$, called del Pezzo surfaces)
- $\dim M \geq 12$: No (J. Fine - D. Panov '10)
- $\dim M = 6, 8, 10$: **Open**

Theorems

Theorem : (Tolman '07 & McDuff '08) Let (M, ω) be a six-dimensional closed monotone symplectic manifold equipped with a Hamiltonian S^1-action. If $b_2(M) = 1$, then there are exactly four such manifolds up to S^1-equivariant symplectomorphism. Moreover, each of them is Fano.

Theorem : (Lindsay - Panov, '17) Let (M, ω) be a six-dimensional closed monotone symplectic manifold equipped with a Hamiltonian S^1-action. Then $\pi_1(M) = 0$.

Theorem : (C., '19) Let (M, ω) be a six-dimensional closed monotone symplectic manifold equipped with a semifree Hamiltonian S^1-action. If a minimal (or maximal) fixed component is of dimension less than or equal to 2, then M is Fano.

Some Basics

Definition : For given symplectic manifold (M, ω), an S^1-action on M is called

- **symplectic** if $t^*\omega = \omega$ for every $t \in S^1$. Equivalently,

$$d i_X \omega = 0$$

- **Hamiltonian** if

$$i_X \omega = -dH \quad \text{for some} \quad H : M \to \mathbb{R}$$

  (H is called **moment map**)

- **semifree** if free outside the fixed point set

Moment maps

Properties of moment map : Assume (M,ω) : Ham. S^1-mfld. with moment map H.

- H : perfect Morse-Bott function. (Every index is even.)
- $z \in M^{S^1}$: fixed point $\Leftrightarrow$ z : critical pt of H.
- Morse index of $z \in M^{S^1}$ = twice the number of negative weights :

  $$T_zM \cong \mathbb{C}^n, \quad t \cdot (z_1, \cdots, z_n) = (t^{\alpha_1}z_1, \cdots, t^{\alpha_n}z_n), \quad \text{ind}(z) = 2 \times (\#\text{ negative } \alpha_i\text{'s})$$
- H is S^1-invariant.

  (Every level set is S^1-invariant $\Rightarrow$ $M_r := H^{-1}(r)/S^1$ called **reduced space**))
- If r is regular, then M_r : orbifold with reduced symplectic form ω_r.
  - For semifree case, M_r is smooth.
  - If $\dim M = 6$, M_c is also smooth for any singular value c.

Moment maps

'Monotonicity' $c_1(TM) = [\omega]$ implies :

- $^\exists$ unique moment map (called **balanced**) such that

$$H(Z) = -\Sigma(Z), \qquad \Sigma(Z) : \text{sum of weights at } Z \subset M^{S^1}$$

(If $\dim M = 6$ and the action is semifree, then

$$H(Z_{\max}) = 3 - \dim Z_{\max}, \quad H(Z_{\min}) = -3 + \dim Z_{\min})$$

- If H is balanced, then (M_0, ω_0) is again monotone such that

$$c_1(TM_0) = [\omega_0]$$

(E.g. If $\dim M = 6$ and the action is semifree, then M_0 : del Pezzo)

Slice Decomposition

Assume that the critical values of $H : M \to \mathbb{R}$ are

$$\min H = c_1 < \cdots < c_k = \max H.$$

Decompose M into a union of Hamiltonian S^1-manifolds with boundaries :

$$N^{2j-1} = H^{-1}(\underbrace{[c_j - \epsilon, c_j + \epsilon]}_{=:I_{2j-1}}), \quad N^{2j} = H^{-1}(\underbrace{[c_j + \epsilon, c_{j+1} - \epsilon]}_{=:I_{2j}})$$

where $\epsilon > 0$ is chosen to be sufficiently small. We call those N_j's **slices**.

Slice Decomposition

- **Regular slice** (N, σ, K, I) **:** free Hamiltonian S^1-manifold (N, σ) with boundary and $K : N \to I = [a, b]$: surjective proper moment map.

- **Critical slice** (N, σ, K, I) **:** semifree Hamiltonian S^1-manifold (N, σ) with boundary and $K : N \to I = [a, b]$ (surj. proper) such that $^\exists$ unique critical value $c \in [a, b]$ satisfying one the followings :
  - (interior slice) $c \in (a, b)$,
  - (maximal slice) $c = b$ and $K^{-1}(c)$ is a critical submanifold,
  - (minimal slice) $c = a$ and $K^{-1}(c)$ is a critical submanifold.

- **Isomorphism of slices :** $^\exists$ S^1-equivariant symplectomorphism ϕ such that

$$\begin{array}{ccc} (N_1, \sigma_1) & \xrightarrow{\ \phi\ } & (N_2, \sigma_2) \\ {\scriptstyle K_1}\downarrow & & \downarrow{\scriptstyle K_2} \\ I_1 & \xrightarrow{\ +k\ } & I_2 \end{array}$$

commutes.

Critical slices

Theorem : (Guillemin-Sternberg '88) Let $(N, \sigma, K, [a, b])$ be an interior critical slice with $c \in [a, b]$. Assume

- the action is semifree,
- the fixed point set $Z_c \subset K^{-1}(c)$: index two (or co-index two, resp.)

Then M_b (M_a, resp.) is the blow-up of M_a (M_b, resp.) along Z_c.

Moreover, if two critical slices $(N_1, \sigma_1, K_1, [a_1, b_1])$ and $(N_2, \sigma_2, K_2, [a_2, b_2])$ satisfy

- $(N_1)_{a_1} \cong (N_2)_{a_2}$,
- $e(K_1^{-1}(a_1)) = e(K_1^{-1}(a_2)) \in H^2((N_1)_{a_1}; \mathbb{Z})$, $\quad$ ($e(\bullet)$: Euler class)
- $^{\exists}$ symplectomorphism $\phi : ((N_1)_{c_1}, (\sigma_1)_{c_1}) \to ((N_2)_{c_2}, (\sigma_2)_{c_2})$ s.t. $\phi(Z_{c_1}) = Z_{c_2}$,

then $^{\exists}\epsilon > 0$ such that

$$(N_1, \sigma_1, K_1, [c_1 - \epsilon, c_1 + \epsilon]) \cong (N_2, \sigma_2, K_2, [c_2 - \epsilon, c_2 + \epsilon])$$

Note : For $\dim M = 6$, the second condition is automatically satisfied.

Regular slices

Question : Which information determines a regular slice uniquely?

Fact : $(N_1, \sigma_1, K_1, [a,b]) \cong (N_2, \sigma_2, K_2, [a,b])$ if and only if

- $e(K_1^{-1}(t)) = e(K_2^{-1}(t))$,
- $(\sigma_1)_t \cong (\sigma_2)_t \quad \forall t \in [a,b]$.

Issue : Extremely hard to check the second condition.

Hope : $[(\sigma_1)_t] = [(\sigma_2)_t]$ implies $(\sigma_1)_t \cong (\sigma_2)_t$. ——————— (*)
(E.g. Moser trick : If $\{\omega_t\}_{t\in[0,1]}$ s.t. $[\omega_t]$: constant, then $\omega_0 \cong \omega_1$)

Symplectic rigidity

Theorem (Gonzalez '09) If every reduced space is *symplectically rigid*, then (*) holds. Moreover, a collection of slices determines a whole manifold uniquely.

Definition : A manifold N is said to be **symplectically rigid** if

- (uniqueness) any two cohomologous symplectic forms are diffeomorphic,
- (deformation implies isotopy) every path ω_t ($t \in [0,1]$) of symplectic forms such that $[\omega_0] = [\omega_1]$ can be homotoped through families of symplectic forms with the fixed endpoints ω_0 and ω_1 to an isotopy, that is, a path ω_t' such that $[\omega_t']$ is constant in $H^2(B)$,
- For every symplectic form ω on N, the group $\mathrm{Symp}(N,\omega)$ of symplectomorphisms that act trivially on $H_*(N;\mathbb{Z})$ is path-connected.

Theorem (Gromov, Abreu-McDuff, Lalonde-Pinsonnault, Pinsonnault)
$\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{P}^2 \# k\overline{\mathbb{P}^2}$ are rigid for $k \leq 3$.

Summing up..

Let (M_i, ω_i) $(i = 1, 2)$: six-dim. closed semifree Ham. S^1-mflds with $c_1(TM_i) = [\omega_i]$. If

- every reduced space is symplectically rigid,
- the fixed point data of (M_i, ω_i)'s coincide,

then their slices in correspondence are isomorphic to each other and are uniquely glued. Hence they are S^1-equivariantly symplectomorphic.

Our strategy :

- Classify all fixed point data and prove symplectic rigidity of reduced spaces.
- For each fixed point data, find the corresponding Fano variety with a semifree $\mathbb{C}^*$-action.

Fixed point data

A **fixed point data** of (M^6, ω, H), denoted by $\mathfrak{F}(M, \omega, H)$, is a collection

$$\mathfrak{F}(M, \omega, H) := \left\{ (M_c, \omega_c, Z_c^1, Z_c^2, \cdots, Z_c^{k_c}, e(P_c^{\pm})) \mid c \in \text{Crit } H \right\}$$

which consists of the information below.

- (M_c, ω_c) is the reduced symplectic manifold at level c.
- k_c is the number of fixed components at level c.
- Each Z_c^i is a connected fixed component and hence is a symplectic submanifold of (M_c, ω_c) via the embedding

$$Z_c^i \hookrightarrow H^{-1}(c) \to H^{-1}(c)/S^1 = M_c.$$

  (This information contains a normal bundle of Z_c^i in M_c.)
- The Euler class $e(P_c^{\pm})$ of principal S^1-bundles $H^{-1}(c \pm \epsilon) \to M_{c \pm \epsilon}$.

Fixed point data

Theorem (Gonzalez '09) Suppose that (M, ω) is a six-dimensional closed semifree Hamiltonian S^1-manifold. If every reduced space is symplectically rigid, then M is determined by the fixed point data up to S^1-equivariant symplectomorphism.

Issue : Difficult to classify embeddings of a submanifold Z_c in M_c.
(Computing homology class $[Z_c] \in H_*(M_c)$ is relatively easy.)

Hope : If $[Z] = [Z'] \in H_*(M_c)$, then $(M_c, Z) \cong (M_c, Z')$.
(This holds in our situation : $M_c \cong \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}^2 \# k\overline{\mathbb{P}^2}$ $(k \leq 3)$ and Z : spheres or an elliptic curve)

Topological fixed point data

A **topological fixed point data** of (M^6, ω, H), denoted by $\mathfrak{F}_{\text{top}}(M, \omega, H)$, is a collection

$$\mathfrak{F}_{\text{top}}(M, \omega, H) := \left\{ (M_c, [\omega_c], [Z_c^1], [Z_c^2], \cdots, [Z_c^{k_c}], e(P_c^{\pm})) \mid c \in \text{Crit } H \right\}$$

where (M_c, ω_c), k_c, Z_c^i, $e(P_c^{\pm})$ are as before.

Note : Much more easier to compute than $\mathfrak{F}(M, \omega, H)$.
(Main technique : adunction equality, localization, D-H theorem..)

Main Theorem

Theorem (C. '19) Suppose that (M, ω) is a six-dimensional closed monotone semifree Hamiltonian S^1-manifold. If one of extremal fixed component is of dimension less than four, then M is S^1-equivariantly symplectomorphic to some Fano 3-fold with some $\mathbb{C}^*$-action.

Sketch of Proof :

- **Step 1 :** Classify all TFD. (They are finitely many.)
  (Consequence : reduced space is either one of $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}^2 \# k\overline{\mathbb{P}^2}$ $(k \leq 3)$.)
- **Step 2 :** Show that TFD determines FD uniquely.
  - Each $((M_c, \omega_c), Z_c)$ is symplectomorphic to an algebraic pair (Siebert-Tian, Zhang).
  - If $H^1(X, \mathcal{O}_X) = 0$, then any smooth curves with a fixed homology class are symplectically isotopic to each other.
- **Step 3 :** In Mori-Mukai's list, we can find a smooth Fano 3-fold for each TFD (as well as $\mathbb{C}^*$-action.)

Main Theorem

	$(M_0, [\omega_0])$	Z_{-3}	Z_{-1}	Z_0	Z_1	Z_2	Z_3	c_1^3
(I-1)	$(\mathbb{C}P^2, 3u)$	pt		$Z_0 \cong S^2$, $[Z_0] = 2u$			pt	54
(I-2)	$(\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}, 3u - E_1 - E_2 - E_3)$	pt	3 pts		3 pts		pt	48
(I-3)	$(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, 3u - E_1)$	pt	pt	$Z_0 = Z_0^1 \dot\cup Z_0^2$ $Z_0^1 \cong Z_0^2 \cong S^2$ $[Z_0^1] = [Z_0^2] = u - E_1$	pt		pt	52
(II-3.1)	$(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, 3u - E_1)$	pt	pt	$Z_0 = S^2$ $[Z_0] = E_1$		S^2		62
(II-3.2)	$(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, 3u - E_1)$	pt	pt	$Z_0 = S^2$ $[Z_0] = u$		S^2		54
(II-3.3)	$(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, 3u - E_1)$	pt	pt	$Z_0 = S^2$ $[Z_0] = 2u - E_1$		S^2		46
(II-4.1)	$(\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}, 3u - E_1 - E_2)$	pt	2 pts	$Z_0 = Z_0^1 \dot\cup Z_0^2$ $Z_0^1 \cong Z_0^2 \cong S^2$ $[Z_0^1] = u - E_1$ $[Z_0^2] = u - E_1 - E_2$	pt	S^2		44
(II-4.2)	$(\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}, 3u - E_1 - E_2 - E_3)$	pt	3 pts	$Z_0 = S^2$ $[Z_0] = u - E_2 - E_3$	2pts	S^2		42
(III-1)	$(\mathbb{C}P^2, 3u)$	pt			$\mathbb{C}P^2$			64
(III-2)	$(\mathbb{C}P^2, 3u)$	pt	pt		$\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$			56
(III-3.1)	$(\mathbb{C}P^2, 3u)$	pt		$Z_0 \cong S^2$, $[Z_0] = u$	$\mathbb{C}P^2$			54
(III-3.2)	$(\mathbb{C}P^2, 3u)$	pt		$Z_0 \cong S^2$, $[Z_0] = 2u$	$\mathbb{C}P^2$			46
(III-3.3)	$(\mathbb{C}P^2, 3u)$	pt		$Z_0 \cong T^2$, $[Z_0] = 3u$	$\mathbb{C}P^2$			40
(III-4.1)	$(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, 3u - E_1)$	pt	pt	$Z_0 \cong S^2$, $[Z_0] = E_1$	$\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$			50
(III-4.2)	$(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, 3u - E_1)$	pt	pt	$Z_0 \cong S^2$, $[Z_0] = u - E_1$	$\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$			50
(III-4.3)	$(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, 3u - E_1)$	pt	pt	$Z_0 \cong S^2$, $[Z_0] = u$	$\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$			46
(III-4.4)	$(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, 3u - E_1)$	pt	pt	$Z_0 \cong S^2$, $[Z_0] = 2u - E_1$	$\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$			42
(III-4.5)	$(\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}, 3u - E_1 - E_2)$	pt	2 pts	$Z_0 \cong S^2$, $[Z_0] = u - E_1 - E_2$	$\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$			46

Example : TFD classification

What we know :

I. $M_0 \cong \mathbb{P}^2,\ \mathbb{P}^1 \times \mathbb{P}^1,\ \mathbb{P}^2 \# \, k\overline{\mathbb{P}^2}$ $(k \leq 8)$, $\quad [\omega_0] = c_1(TM_0)$

II.

$$Z_{\min} = \begin{cases} \text{point} & H(Z_{\min}) = -3 \\ S^2 & H(Z_{\min}) = -2 \\ \dim Z_{\min} = 4 & H(Z_{\min}) = -1 \end{cases}$$

(**Note :** $\pi_1(Z_{\min}) = \pi_1(M_0) = \pi_1(M)$ (Theorem by H. Li))

Example

For $Z_{\min} = Z_{\max} =$ point

——— - Possible interior critical values of H :

$$\{0\}, \quad \{-1,1\}, \quad \{-1,0,1\}$$

Case I : Crit $H = \{-3,0,3\} \quad \Rightarrow \quad Z_{-1} = Z_1 = \emptyset, \quad Z_0 \neq \emptyset.$

- $M_0 \cong \mathbb{P}^2$ (since $H^{-3+\epsilon} \cong S^5$)
- $Z_0 \subset M_0$: (union of) surfaces
- $e(P_{-3+\epsilon}) + \mathrm{PD}(Z_0) = e(P_{3-\epsilon})$

$$\Rightarrow \quad -u + \mathrm{PD}(Z_0) = u$$

$$\Rightarrow \quad Z_0 \cong S^2 \text{ (using adunction formula)}$$

Example

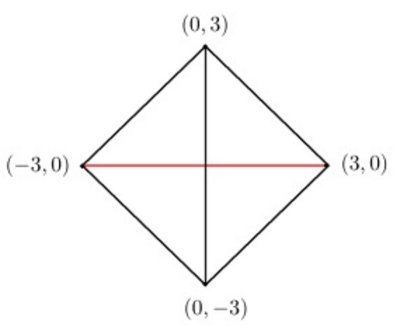


Figure: GKM description of the co-adjoint orbit of $SO(5)$

Example

——- Possible interior critical values of H :

$$\{0\}, \quad \{-1,1\}, \quad \{-1,0,1\}$$

Case II : Crit $H = \{-3,-1,1,3\} \quad \Rightarrow \quad Z_{-1} = Z_1 \neq \emptyset, \quad Z_0 = \emptyset.$

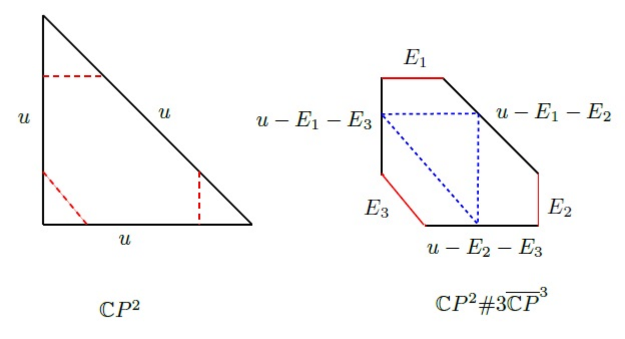
- $Z_{-1} = Z_1 = k$ pts $\quad \Rightarrow \quad M_{-1+\epsilon} \cong \mathbb{P}^2 \# \, k \, \overline{\mathbb{P}^2}$
- $M_0 \cong \mathbb{P}^2 \# \, k \, \overline{\mathbb{P}^2} \quad ([\omega_0] = 3u - E_1 - \cdots - E_k)$
- k blow-downs occur at M_1 where $[\omega_1] = 4u - 2E_1 - \cdots - 2E_k$

$$D \cdot [\omega_1] = 0 \quad \Rightarrow \quad D = u - E_i - E_j \ (i \neq j)$$

where D : exceptional divisor. The number of such D's is k.

- If $k = 1$, $\nexists$ such divisor.
- If $k = 2$, only one such divisor $D = u - E_1 - E_2$.
- If $k = 3$, $\exists$ three divisors.
- If $k > 3$, $\exists$ more than k divisors.

Example



u
u
u
$\mathbb{C}P^2$
E_1
$u - E_1 - E_3$
$u - E_1 - E_2$
E_3
E_2
$u - E_2 - E_3$
$\mathbb{C}P^2 \# 3\overline{\mathbb{C}P}^3$

Example

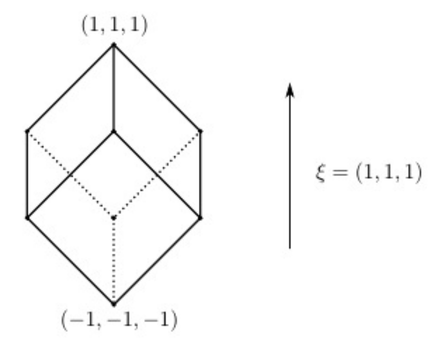


Figure: $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Example

——— - Possible interior critical values of H :

$$\{0\}, \quad \{-1,1\}, \quad \{-1,0,1\}$$

Case III : Crit $H = \{-3,-1,0,1,3\}$.

- $Z_{-1} = Z_1 = k$ pts $\quad \Rightarrow \quad M_{-1+\epsilon} \cong \mathbb{P}^2 \# \, k \, \overline{\mathbb{P}^2}$
- $M_0 \cong \mathbb{P}^2 \# \, k \, \overline{\mathbb{P}^2} \quad ([\omega_0] = 3u - E_1 - \cdots - E_k)$
- Apply ABBV-localization to $c_1^{S^1}(TM)$ & adjunction & k blow-downs at M_1

$\Rightarrow \quad k = 1, \quad \mathrm{PD}(Z_0) = 2u - 2E_1$ (union of two spheres).

Example

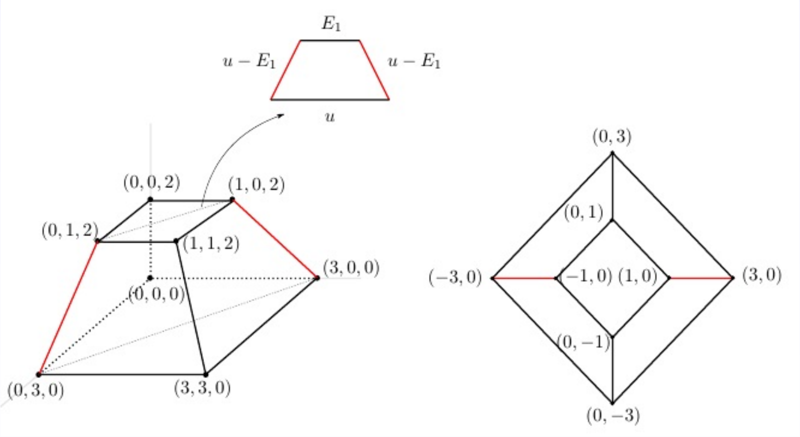


Figure: $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$

Example

Why our (M,ω) is Fano ? Let (M,ω) where the corr. TFD is $\mathfrak{F}(M,\omega)$.

- $^{\exists}$ Fano variety (X,σ,J) with $\mathbb{C}^*$-action having the same TFD $\mathfrak{F}$,

  $\Rightarrow$ Only need to check $(M_0, Z_0, \omega_0) \cong (X_0, Z_0^X, \sigma_0)$
- By symplectic rigidity of M_0, $(M_0,\omega_0) \cong (X_0,\sigma_0)$.
- By symplectic rigidity of M_0, regular slices and gluing data are uniquely determined by $\mathfrak{F}$.
- Use

  - (Siebert-Tian) Any symplectic surface in $\mathbb{P}^2$ of degree ≤ 17 is algebraic.
  - (Zhang) Any symplectic sphere with self-intersection ≥ -1 is algebraic.
  - (Lemma) Any two algebraic curves in M_0 are symplectic isotopic

  (since $H^1(M_0, \mathcal{O}_{M_0}) = 0$.)

  $\Rightarrow$ critical slices are also uniquely determined.