Fundamental groups of small covers

Wu, Lisu

School of Mathematical Sciences, Fudan University

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Content

Small cover

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- The $\mathbb{Z}_2^n$-action on M determines a $\mathbb{Z}_2^n$-valued characteristic function λ on the set of facets of P

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$$\lambda : \mathcal{F}(P) \triangleq \{F_1, F_2, \cdots, F_m\} \longrightarrow \mathbb{Z}_2^n$$
such that
$$\forall\ f = F_1 \cap F_2 \cap \cdots \cap F_k,$$
$$G_f \triangleq \langle \lambda(F_1), \lambda(F_2), \cdots, \lambda(F_k) \rangle \cong \mathbb{Z}_2^k.$$

- Small cover

$$M = P \times \mathbb{Z}_2^n / \sim$$

where $(p, g) \sim (q, h)$ iff $p = q$, $g^{-1}h \in G_{f(p)}$, and $f(p)$ is the unique face of P that contains p in its relative interior, $G_{f(p)} = \{1\}$ if $p \in P^\circ$.

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- Define right-angled Coxeter group of P
$$W = \langle s_F, \forall\, F \in \mathcal{F}(P) \mid s_F^2 = 1; (s_F s_{F'})^2 = 1, F \cap F' \neq \varnothing \rangle$$

- The Borel construction of $\mathbb{Z}_2^n$ on M

$$BP = M \times_{\mathbb{Z}_2^n} E\mathbb{Z}_2^n$$

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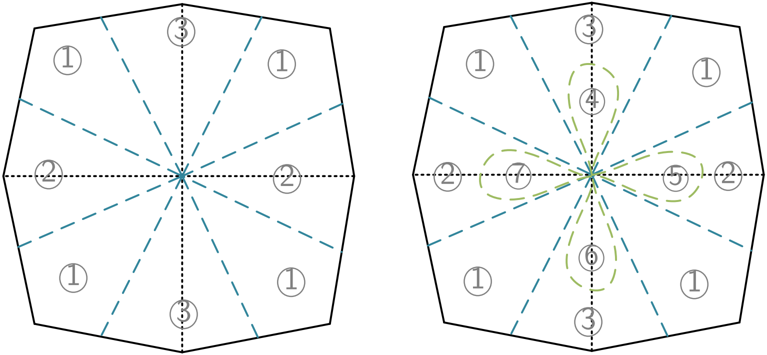
where $E\mathbb{Z}_2^n = (S^\infty)^n$. And $\pi_1(BP) \cong W$.

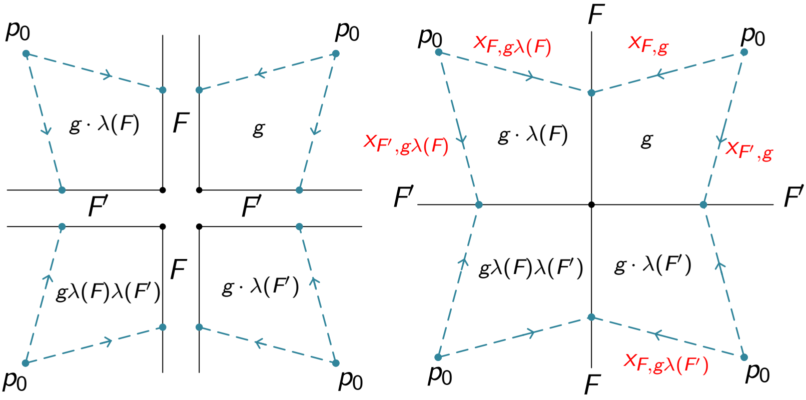
- $M \to BP \to B\mathbb{Z}_2^n$ induces a right-split exact sequence

$$1 \longrightarrow \pi_1(M) \longrightarrow W \underset{\gamma}{\overset{\phi}{\rightleftarrows}} \mathbb{Z}_2^n \longrightarrow 1$$

where $\phi(s_F) = \lambda(F), \ \forall F \in \mathcal{F}(P)$.

Cell decomposition

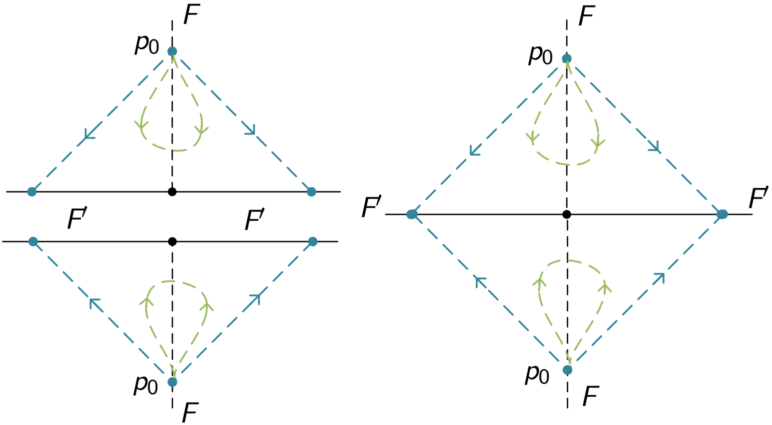




Cell-①

Relation-1: $$x_{F,g}x_{F,g\lambda(F)} = 1$$

Relation-2: $$x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$$



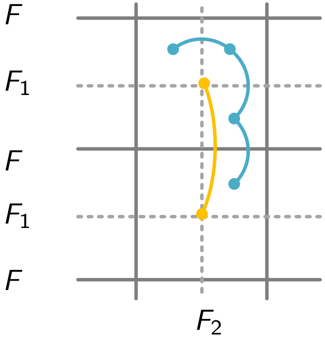
Relation-2: $x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')}$

Relation-3: $x_{F,g} = 1,\ p_0 \subset F$

Presentation of $\pi_1(M, p_0)$

$$\begin{aligned}\pi_1(M,p_0) = \langle x_{F,g}, \forall\, F, g \,| & x_{F,g}x_{F,g\lambda(F)} = 1; \\ & x_{F,g}x_{F',g\lambda(F)} = x_{F',g}x_{F,g\lambda(F')},\ \ F\cap F' \neq \varnothing; \\ & x_{F,g} = 1, p_0 \in F\rangle\end{aligned}$$

$\widetilde{M} = Q \times \pi_1(M)/\sim = P \times W/\sim$



Rk: $\lambda(F_1, F_2, F, F') = (e_1, e_2, e_1 e_2, e_2)$.

Relation between $\pi_1(M)$ and W

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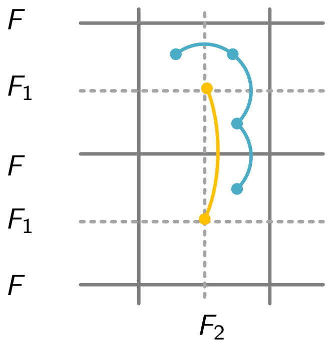

$x_{F,1}(Q,1) \mapsto (Q, x_{F,1})$

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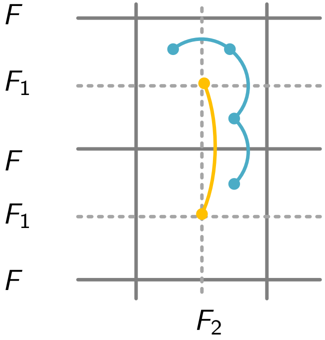

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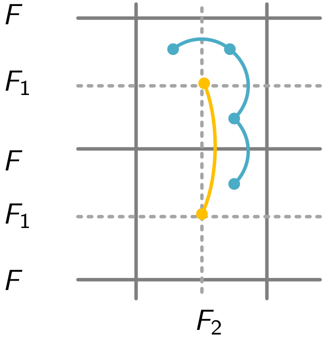

$$x_{F,1}(Q,1) \mapsto (Q, x_{F,1})$$

$$s_F(P,1) \mapsto (P, s_F)$$

$$\begin{tikzcd}
x_{F,1}(P,1) \arrow[r, mapsto] \arrow[d, mapsto, dotted] & (P, s_{F_2}s_{F_1}s_F) \\
\gamma(\lambda(F)) \cdot s_F(P,1) \arrow[r, mapsto] & s_{F_2}s_{F_1}s_F(P,1) \arrow[u, mapsto]
\end{tikzcd}$$

Rk: $\lambda(F_1, F_2, F, F') = (e_1, e_2, e_1e_2, e_2)$.

Relation between $\pi_1(M)$ and W

$$\begin{aligned}
\alpha : \pi_1(M, p_0) &\longrightarrow W \\
x_{F,g} &\longmapsto \quad \gamma(g\lambda(F)) \cdot \gamma(\lambda(F)) s_F \cdot (\gamma(g\lambda(F)))^{-1} \\
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1 \longrightarrow \pi_1(M) \xrightarrow{\alpha} W \xrightarrow{\phi} \mathbb{Z}_2^n \longrightarrow 1 \tag{1}
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Idea

$$\begin{tikzcd} M \arrow[r, dotted] \arrow[d, "\pi"'] & \pi_1(M) \arrow[d, "\alpha"] \\ P \arrow[r, dotted] & W \end{tikzcd}$$

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M \arrow[r, dotted] \arrow[d, "\pi"'] & \pi_1(M) \arrow[d, "\alpha"] \\
\color{red}{P} \arrow[r, dotted, red] & \color{red}{W}
\end{tikzcd}
$$

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\begin{tikzcd}
M \arrow[r, dotted] \arrow[d, "\pi"'] \arrow[dr, phantom, "\begin{array}{c}\text{hard}\\ \downarrow \\ \text{easy}\end{array}"] & \pi_1(M) \arrow[d, "\alpha"] \\
P \arrow[r, dotted] & W
\end{tikzcd}
$$

For any proper face f of P,

- Define $\mathcal{F}(f^{\perp}) \overset{\triangle}{=} \{F \in \mathcal{F}(P) \mid \dim(f \cap F) = \dim(f) - 1\}$. So $\mathcal{F}(f^{\perp})$ consists of those facets of P that intersect f transversely.

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Theorem (Wu-Yu, 2017)

Let M be a small cover over a simple polytope P and f be a proper face of P. The following two statements are equivalent.

> *The facial submanifold M_f is π_1-injective.*

> *For any $F, F' \in \mathcal{F}(f^{\perp})$, we have $f \cap F \cap F' \neq \varnothing$ whenever $F \cap F' \neq \varnothing$.*

Flag cases

A simple polytope P is called flag if a collection of facets of P has common intersection whenever they pairwise intersect.

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Proposition (Wu-Yu, 2017)

Let M be a small cover over P. Then P is flag if and only if every facial submanifold of M is π_1-injective.

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- A *k-circuit* in P is a simple loop on the boundary of P which intersects transversely with the interior of exactly k distinct edges, and a k-circuit is called *prismatic* if the endpoints of those edges are distinct.
- A *k-belt* in P is a set of k distinct faces $F_1, \cdots, F_k$ of P such that $F_i \cap F_{i+1} \neq \varnothing$ for $1 \leq i \leq k-1$, $F_k \cap F_1 \neq \varnothing$, and any three faces in the belt have no common intersection.

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- Each k-belt can determine a prismatic k-circuit. A prismatic 3-circuit determines a 3-belt; and if there is no prismatic 3-circuit, then a prismatic 4-circuit determines a 4-belt.

Let M be a 3-small cover over $P(\neq \Delta^3)$, the following facts was referred in DJ's paper(corrected):

- If there exist prismatic 3-circuits in P, then M can be decomposed into prime pieces glued along S^2 or $\mathbb{R}P^2$.

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- If there is no prismatic 3-circuit but prismatic 4-circuits in P, then M can be decomposed into atoroidal and Seifert fibered pieces glued along tori or Klein bottles.
- If there is no prismatic 3 or 4-circuit in P, then M is hyperbolic.

Prime decomposition theorem

Let M be a connected 3-manifold.

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Theorem (Kneser, Milnor, Prime Decomposition Theorem)

Each compact 3-manifold M can factor as a connected sum of prime manifolds. This decomposition is unique under the assumption that M is orientable.

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In particular, the prime decomposition of an oriented 3-small cover is equivalent to cutting surgery along prismatic 3-circuits in P.

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Theorem (Jaco-Shalen-Johannson, Torus Decomposition Theorem)

For an oriented, irreducible, closed 3-manifold, there exists a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \cdots, T_m$ such that each component of M cut along $T_1 \cup \cdots \cup T_m$ is atoroidal or Seifert fibered, and a minimal such collection $T_1, \cdots, T_m$ is unique up to isotopy.

Theorem (Perelman, Geometrization Theorem)

Let M be a irreducible colsed 3-manifold. There exists a (possibly empty) collection of disjointly embedded incompressible surface $S_1, \cdots, S_m$ which are either tori or Klein bottles, such that each component of M cut along $S_1 \cup \cdots \cup S_m$ is geometric. Any such collection of tori with a minimal number of components is unique up to isotopy.

JSJ-decomposition theorem

Theorem (Perelman, Geometrization Theorem)

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Proposition

Let M be a 3-small cover over a simple polytope P, then M is atoroidal if and only if there is no 4-belt in P.
In particular, the JSJ decomposition or geometric decomposition of a irreducible 3-small cover is equivalent to cutting surgery along prismatic 4-circuits in P.

Theorem (Thurston, Hyperbolization Theorem)

Each irreducible, atoroidal, closed 3-manifold with infinite fundamental group is hyperbolic.

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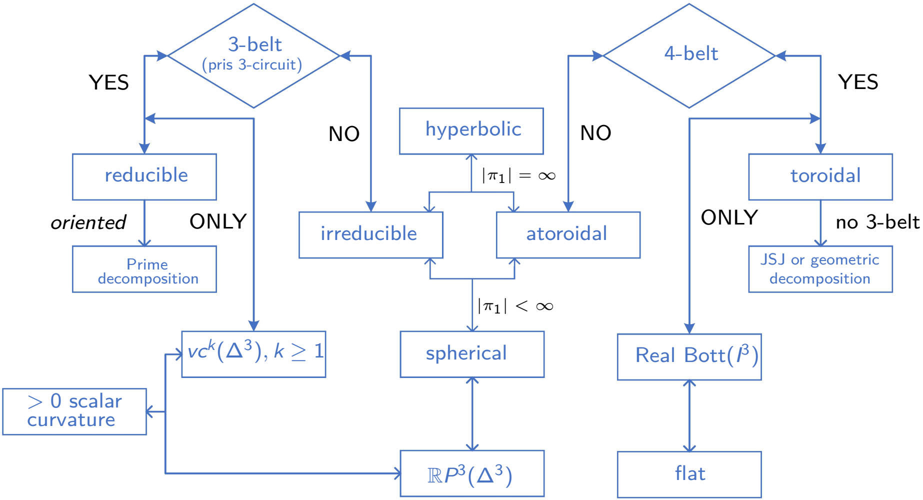
Each irreducible, atoroidal, closed 3-manifold with infinite fundamental group is hyperbolic.

Proposition

Let M be a 3-small cover over a simple polytope $P(\neq \Delta^3)$, then M is hyperbolic if and only if there is no prismatic 3 or 4-circuit in P.

Proposition (Wu-Yu, 2017)

A small cover M over a simple 3-polytope P can admit a Riemannian metric with nonnegative scalar curvature if and only if P is combinatorially equivalent to the cube $[0,1]^3$ or a polytope obtained from Δ^3 by a sequence of vertex cuts.


3-belt
(pris 3-circuit)
YES
NO
reducible
oriented
ONLY
Prime decomposition
irreducible
hyperbolic
$|\pi_1| = \infty$
atoroidal
$|\pi_1| < \infty$
spherical
$vc^k(\Delta^3), k \geq 1$
> 0 scalar curvature
$\mathbb{R}P^3(\Delta^3)$
4-belt
NO
YES
toroidal
ONLY
no 3-belt
JSJ or geometric decomposition
Real Bott(I^3)
flat

End of Talk

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Some references

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Email: wulisuwulisu@qq.com
Homepage: http://algebraic.top/