Towards transverse toric geometry

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The main concerns:

1. complex manifold M with a maximal action of compact torus,
2. canonical foliation F on M.

Purpose

The foliated manifold (M, F) behaves similar to a toric variety.

M : connected manifold,
G : compact torus, $G \curvearrowright M$: effective.

For $x \in M$, T_xM is a faithful representation of G_x.

$$T_xM \cong \underbrace{T_x(G \cdot x)}_{\text{trivial}} \oplus \underbrace{T_xM/T_x(G \cdot x)}_{\text{faithful}}.$$

In particular, $G_x \to GL(T_xM/T_x(G \cdot x))$ is injective.

$$\dim G_x = \dim G_x^0 \leq \frac{1}{2}(\dim M - \dim G \cdot x).$$

$$\Longrightarrow \dim G + \dim G_x \leq \dim M.$$

Example

${}^{\exists}x \in M^G \Longrightarrow \dim G \leq \frac{1}{2} \dim M.$

Definition

M : connected manifold,

G : compact torus, $G \curvearrowright M$: effective.

$G \curvearrowright M$ is maximal if

$$^{\exists}x \in M \text{ s.t. } \dim G + \dim G_x = \dim M.$$

Remark

$G \curvearrowright M$ maximal $\Longrightarrow$ G is a maximal compact torus in $\mathrm{Diff}\, M$.

M : compact connected complex manifold,
$G \subset \mathrm{Aut}(M, J)$: maximal compact torus.

$\mathfrak{g} \subset \mathfrak{X}(M, J)$. Put

$$\mathfrak{h}' := \mathfrak{g} \cap J\mathfrak{g} \subset \mathfrak{g}, \quad H' := \exp \mathfrak{h}'.$$

Proposition

The followings hold:

1. H' is a complex Lie group.
2. $H' \curvearrowright M$: holomorphic and local free.
3. H' does not depend on the choice of G.

Definition

The canonical foliation F on M is a foliation whose leaves are H'-orbits.

Example (complex tori)

- $M = \mathbb{C}^n/\Gamma$, Γ : $\mathbb{C}^n$ lattice.
- $G = M = \mathbb{C}^n/\Gamma \cong (S^1)^{2n}$.
- For all $x \in M$,

$$\underbrace{\dim G}_{2n} + \underbrace{\dim G_x}_{0} = \underbrace{\dim M}_{2n}.$$

- $\mathfrak{h}' = \mathfrak{g} \cap J\mathfrak{g} = \mathfrak{g}$ because any fundamental vector field is J of a fundamental vector field.
- The leaf space M/F is a point, i.e. 0-dimensional toric variety.

Example (toric varieties)

- M : nonsingular complete toric variety of dimension n,
- $G = (S^1)^n \subset (\mathbb{C}^\times)^n \curvearrowright M$.
- For $x \in M^G$,

$$\underbrace{\dim G}_{n} + \underbrace{\dim G_x}_{n} = \underbrace{\dim M}_{2n}.$$

- $\mathfrak{h}' = \mathfrak{g} \cap J\mathfrak{g} = \{0\}$, i.e. every leaf of F is a point.
- The leaf space M/F is nothing but a toric variety M.

Example (Hopf surfaces)

- $\alpha_1, \alpha_2 \in \mathbb{C}$, $1 < |\alpha_1| \leq |\alpha_2|$.
- $\Gamma = \{(\alpha_1^k, \alpha_2^k) \in (\mathbb{C}^\times)^2 \mid k \in \mathbb{Z}\} \subset (\mathbb{C}^\times)^2$. $\Gamma \cong \mathbb{Z}$.
- $(\mathbb{C}^\times)^2 \curvearrowright \mathbb{C}^2 \setminus \{0\}$ via $(g_1, g_2) \cdot (z_1, z_2) = (g_1 z_1, g_2 z_2)$.
- $M_{\alpha_1,\alpha_2} := (\mathbb{C}^2 \setminus \{0\})/\Gamma$ Hopf surface.
- G : maximal compact torus in $(\mathbb{C}^\times)^2/\Gamma \curvearrowright M$. $G \cong (S^1)^3$.
- For $x = [1, 0] \in M_{\alpha_1,\alpha_2}$,

$$\underbrace{\dim G}_{3} + \underbrace{\dim G_x}_{1} = \underbrace{\dim M_{\alpha_1,\alpha_2}}_{4}.$$

- $\mathfrak{h}' \subset \mathfrak{g}$ is 2-dimensional subspace.
  $H' \subset G$ is a subtorus iff $\alpha_1^{n_1} = \alpha_2^{n_2}$ for some $n_1, n_2 \in \mathbb{N}$.
- M/F is Hausdorff iff $\alpha_1^{n_1} = \alpha_2^{n_2}$ for some $n_1, n_2 \in \mathbb{N}$.

$\mathcal{C}_1$: category of compact connected complex manifolds with maximal actions.

- An object of $\mathcal{C}_1$ is (M, G, y):
  - G : compact torus,
  - M : compact connected complex manifold, $G \curvearrowright M$ maximal, preserving the complex structure.
  - $y \in M$ such that $G_y = \{1_G\}$.
- For $(M_1, G_1, y_1), (M_2, G_2, y_2)$, $(f, \alpha) \in \mathrm{Hom}_{\mathcal{C}_1}((M_1, G_1, y_1), (M_2, G_2, y_2))$ if
  - $\alpha\colon\ G_1 \to G_2$ is a smooth homomorphism,
  - $f\colon\ M_1 \to M_2$ is an α-equivariant holomorphic map,
  - $f(y_1) = y_2$.

$\mathcal{C}_2$: category.

- An object of $\mathcal{C}_2$ is $(\Delta, \mathfrak{h}, G)$:
    - G is a compact torus,
    - Δ is a nonsingular fan in $\mathfrak{g}$ with respect to the lattice $\ker \exp_G$,
    - $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ is a $\mathbb{C}$-subspace

  satisfying
    - For the projection $p\colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}$, $p|_{\mathfrak{h}}\colon \mathfrak{h} \to \mathfrak{g}$ is injective.
    - $\mathfrak{h}' := p(\mathfrak{h})$. The quotient map $q\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}'$ sends Δ to a complete fan in $\mathfrak{g}/\mathfrak{h}'$.
- For $(\Delta_1, \mathfrak{h}_1, G_1), (\Delta_2, \mathfrak{h}_2, G_2)$, $\alpha \in \mathrm{Hom}_{\mathcal{C}_2}((\Delta_1, \mathfrak{h}_1, G_1), (\Delta_2, \mathfrak{h}_2, G_2))$ if
    - $\alpha\colon G_1 \to G_2$ is a smooth homomorphism,
    - $d\alpha\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ induces a morphism of fans $d\alpha\colon \Delta_1 \to \Delta_2$,
    - $d\alpha^{\mathbb{C}}\colon \mathfrak{g}_1^{\mathbb{C}} \to \mathfrak{g}_2^{\mathbb{C}}$ satisfies $d\alpha^{\mathbb{C}}(\mathfrak{h}_1) \subset \mathfrak{h}_2$.

- For $(M, G, y) \in \mathcal{C}_1$, we can
  - get a nonsingular fan Δ by an argument similar to toric geometry.
  - get a $\mathbb{C}$-subspace $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}}$. $H \subset G^{\mathbb{C}} \curvearrowright M$: global stabilizers.
  - verify $(\Delta, \mathfrak{h}, G) \in \mathcal{C}_2$.
- For $(\Delta, \mathfrak{h}, G) \in \mathcal{C}_2$,
  - $X(\Delta)/H$ is a compact connected complex manifold.
  - $G \curvearrowright X(\Delta)$ descends to a maximal action $G \curvearrowright X(\Delta)/H$.

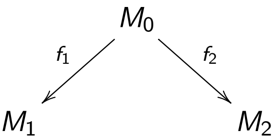
Theorem

$\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent as categories.

Definition

Foliated manifolds (M_1, F_1) and (M_2, F_2) are transverse equivalent if there exist a foliated manifold (M_0, F_0) and (holomorphic or smooth) maps $f_i\colon\ M_0 \to M_i$ such that

1. f_i is a surjective submersion,
2. $f_i^{-1}(x_i)$ is connected for all $x_i \in M_i$,
3. the preimage of each leaf of F_0 by f_i is a leaf of F_i.



Problem

$(M_1, G_1, y_1), (M_2, G_2, y_2) \in \mathcal{C}_1$.
F_i : canonical foliation on M_i, $i = 1, 2$.
When are (M_1, F_1) and (M_2, F_2) transversely equivalent?

Definition

$(M_1, G_1, y_1), (M_2, G_2, y_2) \in \mathcal{C}_1$ are principal equivalent if there exists $(M_0, G_0, y_0) \in \mathcal{C}_1$ and morphisms $(f_i, \alpha_i) \in \mathrm{Hom}_{\mathcal{C}_1}((M_0, G_0, y_0), (M_i, G_i, y_i))$ such that

1. $f_i \colon M_0 \to M_i$ is a principal $\ker \alpha_i$-bundle,
2. $\ker \alpha_i$ is connected.

Proposition (sufficient condition)

If (M_1, G_1, y_1) and (M_2, G_2, y_2) are principal equivalent, then (M_1, F_1) and (M_2, F_2) are transversely equivalent.

Definition

A marked fan is a quadruple $(\widetilde{V}, \widetilde{\Gamma}, \widetilde{\Delta}, \widetilde{\lambda})$, where

- $\widetilde{V}$ is a finite dimensional $\mathbb{R}$-vector space;
- $\widetilde{\Gamma}$ is a finitely generated subgroup of $\widetilde{V}$ that spans $\widetilde{V}$ $\mathbb{R}$-linearly;
- $\widetilde{\Delta}$ is a fan in $\widetilde{V}$ and each 1-cone is generated by an element of $\widetilde{\Gamma}$;
- $\widetilde{\lambda}$ is a function $\widetilde{\lambda}\colon\ \widetilde{\Delta}^{(1)} \to \widetilde{\Gamma}$, where $\widetilde{\lambda}(\rho)$ is a generator of $\rho \in \widetilde{\Delta}^{(1)}$.

$(\widetilde{V}, \widetilde{\Gamma}, \widetilde{\Delta}, \widetilde{\lambda})$: simplicial $\underset{\text{def.}}{\Longleftrightarrow}$ $\widetilde{\Delta}$: simplicial.

$(\widetilde{V}, \widetilde{\Gamma}, \widetilde{\Delta}, \widetilde{\lambda})$: complete $\underset{\text{def.}}{\Longleftrightarrow}$ $\widetilde{\Delta}$: complete.

$\widetilde{\mathcal{C}}_2$: class of all complete simplicial marked fans.

To $(M, G, y) \in \mathcal{C}_1$, we can assign $(\widetilde{V}, \widetilde{\Gamma}, \widetilde{\Delta}, \widetilde{\lambda}) \in \widetilde{\mathcal{C}}_2$. Let $(\Delta, \mathfrak{h}, G) \in \mathcal{C}_2$ be the counterpart to (M, G, y).

- $\widetilde{V} := \mathfrak{g}/p(\mathfrak{h})$,
- $\widetilde{\Gamma} := q(\ker \exp_G) \subset \widetilde{V}$,
- $\widetilde{\Delta} := q(\Delta)$,
- $\widetilde{\lambda} = q \circ \lambda$.

We get $\widetilde{\mathcal{F}}_1 \colon \mathcal{C}_1 \to \widetilde{\mathcal{C}}_2$.

Definition

$(\widetilde{V}_1, \widetilde{\Gamma}_1, \widetilde{\Delta}_1, \widetilde{\lambda}_1)$ and $(\widetilde{V}_2, \widetilde{\Gamma}_2, \widetilde{\Delta}_2, \widetilde{\lambda}_2)$ are isomorphic
$\underset{\text{def.}}{\Longleftrightarrow} {}^{\exists}\varphi\colon\ \widetilde{V}_1 \to \widetilde{V}_2$ linear isomorphism s.t.

1. $\varphi(\widetilde{\Gamma}_1) = \widetilde{\Gamma}_2$,
2. $\varphi(\widetilde{\Delta}_1) = \widetilde{\Delta}_2$,
3. $\varphi \circ \lambda_1 = \lambda_2 \circ \varphi$.

Example (Hopf surface)

M_{α_1,α_2} : Hopf surface.
The marked fan corresponding to M_{α_1,α_2} is isomorphic to $(\widetilde{V}, \widetilde{\Gamma}, \widetilde{\Delta}, \widetilde{\lambda})$, where $\widetilde{V} = \mathbb{R}$,
$\widetilde{\Gamma} = \langle \log|\alpha_1|, \log|\alpha_2|, \log|\alpha_1|\frac{\arg\alpha_2}{2\pi} - \log|\alpha_2|\frac{\arg\alpha_1}{2\pi}\rangle, \ldots$

Theorem

$(M_1, G_1, y_1), (M_2, G_2, y_2) \in \mathcal{C}_1$ are principal equivalent iff $\widetilde{\mathcal{F}}_1(M_1, G_1, y_1)$ and $\widetilde{\mathcal{F}}_1(M_2, G_2, y_2)$ are isomorphic.

Theorem

$\widetilde{\mathcal{F}}_1 \colon \mathcal{C}_1 \to \widetilde{\mathcal{C}}_2$ is essentially surjective.
Namely, ${}^\forall(\widetilde{V}, \widetilde{\Gamma}, \widetilde{\Delta}, \widetilde{\lambda}) \in \widetilde{\mathcal{C}}_2, {}^\exists(M, G, y) \in \mathcal{C}_1$ s.t. $\widetilde{\mathcal{F}}_1(M, G, y)$ is isomorphic to $(\widetilde{V}, \widetilde{\Gamma}, \widetilde{\Delta}, \widetilde{\lambda})$.

principal equivalence $\Longleftrightarrow$ marked fan isomorphism

$\Downarrow$

transverse equivalence

$(M, G, y) \in \mathcal{C}_1$, $(\widetilde{V}, \widetilde{\Gamma}, \widetilde{\Delta}, \widetilde{\lambda}) = \widetilde{\mathcal{F}}_1(M, G, y)$.

Theorem

The leaf space M/F is a toric orbifold iff $\operatorname{rk} \widetilde{\Gamma} = \dim \widetilde{V}$.

Theorem

M is transverse Kähler w.r.t. F iff $\widetilde{\Delta}$ is polytopal.

Theorem

$$H_B^*(M) \cong \mathbb{R}[x_1, \dots, x_m]/\mathcal{I} + \mathcal{J},$$

$\mathcal{I}$: Stanley-Reisner ideal for the underlying simplicial cpx. of $\widetilde{\Delta}$,
$\mathcal{J}$: linear ideal determined by $\widetilde{\lambda}$.