Equivariant Cohomology of Torus Orbifolds

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Joint work with S. Kuroki and J. Song

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Table of contents

Toric and Quasitoric Manifolds

Toric Manifolds	$\longleftrightarrow$	comp. reg. fans
$(\mathbb{C}^*)^n \circlearrowright X^{2n}$	$\longleftrightarrow$	$\Sigma \subseteq \mathbb{R}^n$
comp. non-sing. toric varieties		

Toric and Quasitoric Manifolds

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Quasitoric Manifolds	$\longleftrightarrow$	Characteristic Pairs
$T^n \circlearrowright M^{2n}$	$\longleftrightarrow$	(P, λ)
loc. std. st. $M/T^n \cong P^n$		$\lambda\colon \mathcal{F}(P) \to \mathbb{Z}^n$
P is a simple polytope		satisfies basis condition

Equivariant Cohmology

Set $T := T^n = S^1 \times \cdots \times S^1$ and let

$$ET \longrightarrow BT$$

be the universal T-bundle.

If $T \circlearrowright M$, then we get a fibration

$$M \longrightarrow ET \times_T M \longrightarrow BT.$$

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Definition (Equivariant Cohomology)

$$H_T^* M := H^*(ET \times_T M)$$

Equivariant Cohomology

The equivariant cohomology of toric and quasitoric manifolds can be written in terms of the *face rings* of the corresponding combinatorial objects.

Theorem (Danilov '78 & Jurkiewicz '85)

$$H_T^*(X(\Sigma)) \cong \mathbb{Z}[\Sigma]$$

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Their non-equivariant cohomology rings,

$$H^*(X(\Sigma)) \cong \mathbb{Z}[\Sigma]/\mathcal{J}_\Sigma \quad \& \quad H^*(M(P,\lambda)) \cong \mathbb{Z}[P]/\mathcal{J}_\lambda,$$

are given by factoring out by linear relations.

Properties

- Torus actions are locally standard
- Quotient M/T is a *manifold with corners* (locally like $\mathbb{R}^n_{\geq}$)
- Isolated fixed points
- Can be rebuilt using combinatorial data: $T \times P/\sim$
- Cohomology is generated in degree 2 and hence concentrated in even degrees
- No torsion

$\mathbb{C}P^n = M(\Delta^n, (I_n \mid -\mathbf{1}))$

Example

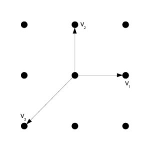

Figure: The fan of $\mathbb{C}P^2$

$$H_T^*(\mathbb{C}P^2(\Sigma)) \cong \mathbb{Z}[v_1, v_2, v_3]/(v_1 v_2 v_3) \cong H_T^*(M(\Delta^2, \lambda))$$

$$H^*(\mathbb{C}P^2(\Sigma)) \cong \mathbb{Z}[v_1, v_2, v_3]/(v_1 v_2 v_3, v_1 - v_3, v_2 - v_3) \cong \mathbb{Z}[v]/(v^3)$$

Torus Manifolds

Definition

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If the T-action is locally standard then the quotient $Q := M/T$ is a manifold with corners.

Torus Graphs

Let M be a torus manifold. For $p \in M^T$

$$T_pM \;\cong\; V_1(p) \oplus \cdots \oplus V_n(p), \tag{1}$$

where $V_i(p) \in \mathrm{Hom}(T, S^1) \cong \mathbb{Z}^n$ is a complex 1-dim T-representation.

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$$\mathcal{S}_M := \{\text{2-dim submflds of } M \\ \text{each fixed ptwise by a codim-1 subtorus of } T\}.$$

Every $S \in \mathcal{S}_M$ is diffeo to a 2-sphere and contains exactly two T-fixed points.

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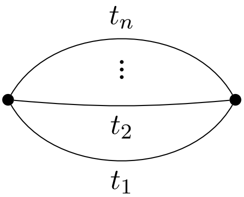
Define an n-valent graph Γ_M whose vertex set is M^T and whose edges correspond to the 2-spheres from $\mathcal{S}_M$. Label each oriented edge with its corresponding weight from (1).

Example S^{2n}

Taking the torus manifolds S^{2n}, where the torus action is obtained by suspending the standard coordinatewise torus action on S^{2n-1}, we obtain a graph with two vertices and n edges between them.

The edges are labelled by the standard basis vectors $\{t_1, \ldots, t_n\}$ of

$$H^2 BT \cong \mathrm{Hom}(T, S^1) \cong \mathbb{Z}^n.$$

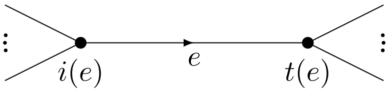


Obviously these are not toric or quasitoric manifolds for $n > 1$.

Abstract Torus Graphs

Let

- Γ be an n-valent connected graph with $n \geq 1$.
- $\mathcal{V}(\Gamma)$ denote the set of vertices.
- $\mathcal{E}(\Gamma)$ denote the set of *oriented* edges.



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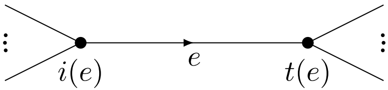

For $p \in \mathcal{V}(\Gamma)$, define

$$\mathcal{E}(\Gamma)_p := \{e \in \mathcal{E}(\Gamma) \mid i(e) = p\}.$$

Abstract Torus Graphs

Definition (Axial Function)

An *axial function* is a map

$$\alpha\colon \mathcal{E}(\Gamma) \longrightarrow \mathrm{Hom}(T, S^1) \cong \mathbb{Z}^n,$$

satisfying the following conditions:

1. $\alpha(\bar{e}) = \pm\alpha(e)$;
2. elements of $\alpha(\mathcal{E}(\Gamma)_p)$ form a basis of $\mathbb{Z}^n$;
3. $\alpha(\mathcal{E}(\Gamma)_{t(e)}) \equiv \alpha(\mathcal{E}(\Gamma)_{i(e)}) \bmod \alpha(e)$, for any $e \in \mathcal{E}(\Gamma)$.

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Definition (Torus Graph)

A *torus graph* is a pair (Γ, α) consisting of an n-valent graph Γ with an axial function α.

$H^{\text{odd}} M = 0$

$$\begin{aligned} H^{\text{odd}} M = 0 &\implies T\text{-action is locally standard} \\ &\implies Q \text{ is a manifold with corners} \end{aligned}$$

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$$\begin{aligned} H^{\mathrm{odd}}M \;=\; 0 &\iff H_T^*M \;\cong\; H^*M \otimes H^*BT, \text{ as } H^*BT\text{-modules} \\ &\implies H_T^*M \text{ is a free } H^*BT\text{-module} \\ &\implies \text{Serre SS of } M \to ET \times_T M \to BT \text{ collapses} \\ &\implies i^*\colon H_T^*M \longrightarrow H_T^*M^T \cong \bigoplus_{p \in M^T} H^*BT \text{ is injective} \end{aligned}$$

How can we describe the image of i^*?

Graph Cohomology (Piecewise Polynomials)

$$H_T^*\Gamma := \{f\colon \mathcal{V}(\Gamma) \to H^*BT \mid f(i(e)) \equiv f(t(e)) \bmod \alpha(e)\}$$

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Theorem (GKM, Masuda & Panov)

If $H^{odd}M = 0$, *then*

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Proof (Sketch).

All isotropy subgroups are connected. Hence the *Chang-Skjelbred* sequence

$$0 \longrightarrow H_T^*M \xrightarrow{i^*} H_T^*M_0 \xrightarrow{\delta} H_T^{*+1}(M_1, M_0) \longrightarrow \cdots$$

is exact with integer coefficients [Franz & Puppe '07], where M_0 and M_1 denote the set of fixed points and 1-dim orbits in M resp.

Graph Cohomology (Piecewise Polynomials)

Proof (Sketch).

So

$$H_T^* M \cong \operatorname{Ker} \delta,$$

which we can rewrite as

$$\delta \colon \bigoplus_{p \in M^T} H^* BT \longrightarrow \bigoplus_{e \in \mathcal{E}(\Gamma)} H^* BT_e,$$

where $T_e := \operatorname{Ker} \alpha(e) \cong T^{n-1}$, and is defined by

$$\delta(\{f(p)\}_{p \in M^T}) = \{f(i(e))|_{H^*(BT_e)} - f(t(e))|_{H^*(BT_e)}\}_{e \in \mathcal{E}(\Gamma)}.$$

The result follows. □

Thom Classes

For any k-dim *face* F of (Γ, α) we define the *Thom class* of F as a map

$$\tau_F \colon \mathcal{V}(\Gamma) \longrightarrow H^{2(n-k)}BT$$

$$\tau_F(p) := \begin{cases} \prod_{i(e)=p,\ e \notin F} \alpha(e), & \text{if } p \in \mathcal{V}(F); \\ 0, & \text{otherwise.} \end{cases}$$

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Lemma

$$\tau_F \in H_T^*\Gamma \quad \& \quad \tau_G \tau_H = \tau_{G \vee H} \cdot \sum_{E \in G \cap H} \tau_E.$$

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Theorem (Masuda & Panov '06)

$$H_T^*\Gamma \cong \mathbb{Z}[\tau_F \mid F \text{ a face}]/\mathcal{I},$$

where $\mathcal{I} = \langle \tau_G \tau_H - \tau_{G \vee H} \cdot \sum_{E \in G \cap H} \tau_E \rangle$.

Toric & Quasitoric Orbifolds

Orbifold – locally like $\mathbb{R}^n/G$ for some finite group G.

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loc. std. st. $X/T \cong P$		$\lambda \colon \mathcal{F}(P) \to \mathbb{Z}^n$
P is a simple polytope		satisfies linear independence condition

Weighted Projective Space

Given a *weight vector* $\chi = (\chi_0, \ldots, \chi_n) \in \mathbb{N}^{n+1}$, define

$$\mathbb{P}(\chi) := S^{2n+1}/S^1\langle\chi\rangle,$$

where $t \cdot (z_0, \ldots, z_n) = (t^{\chi_0} z_0, \ldots, t^{\chi_n} z_n)$.
Weighted projective spaces are examples of (quasi)toric orbifolds over the simplex.

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Theorem (Kawasaki '73)

$$H^i\mathbb{P}(\chi) \cong \begin{cases} \mathbb{Z}, & 0 \leq i = 2j \leq 2n; \\ 0, & \textit{otherwise.} \end{cases}$$

with a twisted product structure

$$\gamma_i \cup \gamma_j = \frac{\ell_i \ell_j}{\ell_{i+j}} \gamma_{i+j}.$$

Equivariant Cohomology

Theorem (Poddar & Sarkar '10)

$$H_T^*(X(\Sigma); \mathbb{Q}) \cong \mathbb{Q}[\Sigma]$$

$$H_T^*(X(P, \lambda); \mathbb{Q}) \cong \mathbb{Q}[P]$$

What about integer coefficients?

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Definition (Rational Axial Function)

A *rational axial function* is a map

$$\alpha \colon \mathcal{E}(\Gamma) \longrightarrow H^2(BT;\mathbb{Q}) \cong \mathbb{Q}^n,$$

satisfying the following conditions:

1. $\forall e \in \mathcal{E}(\Gamma), \exists r_e, r_{\bar{e}} \in \mathbb{Z}$ st. $r_e\alpha(e) = \pm r_{\bar{e}}\alpha(\bar{e}) \in H^2(BT;\mathbb{Z})$;
2. elements of $\alpha(\mathcal{E}(\Gamma)_p)$ are linearly independent in $\mathbb{Q}^n$;
3. $\alpha(\mathcal{E}(\Gamma)_{t(e)}) \equiv \alpha(\mathcal{E}(\Gamma)_{i(e)})$ mod $r_e\alpha(e)$, for any $e \in \mathcal{E}(\Gamma)$.

Graph Cohomology

Given a *rational torus graph* (Γ, α) we define its cohomology as follows:

$$H_T^*\Gamma := \{f\colon \mathcal{V}(\Gamma) \to H^*(BT;\mathbb{Z}) \mid f(i(e)) \equiv f(t(e)) \bmod r_e\alpha(e)\},$$

$$H_T^*(\Gamma;\mathbb{Q}) := \{f\colon \mathcal{V}(\Gamma) \to H^*(BT;\mathbb{Q}) \mid f(i(e)) \equiv f(t(e)) \bmod \alpha(e)\}.$$

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Theorem (D., Kuroki & Song)

Let X be a torus orbifold. Then

$$H^{odd}X = 0 \implies H_T^*X \cong H_T^*\Gamma$$

as $H^*(BT)$-algebras.

Graph Cohomology

Define the map

$$\varphi\colon \mathbb{Q}\left[x_F \mid F \text{ a face}\right] \longrightarrow H_T^*(\Gamma;\mathbb{Q})$$
$$x_F \longmapsto \tau_F$$

and consider the following subring of $\mathbb{Q}[x_F \mid F \text{ a face}]$:

$$\mathbb{Z}\{\Gamma\} := \{f \in \mathbb{Q}[x_F \mid F \text{ a face}] \mid \forall v \in \mathcal{V}(\Gamma), \varphi(f)|_v \in H^*(BT;\mathbb{Z})\}.$$

This set is closed under the addition and multiplication induced from $\mathbb{Q}[x_F \mid F \text{ a face}]$.

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where $\mathcal{J}$ is given by linear relations that can be read off from the combinatorial data.

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This has been done for the limited case of projective toric orbifolds by Bhari, Sarkar & Song '17.

Thank you!