

Asymptotic Theory of Groups

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Assume known: the concept of a [GROUP](#).

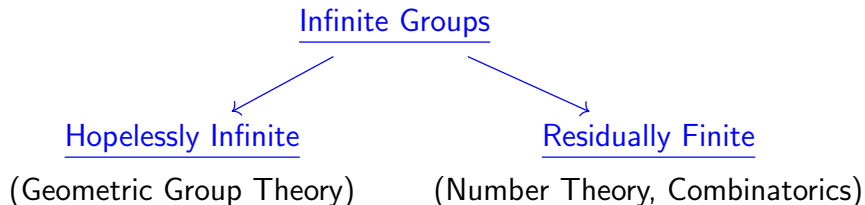
G a group, $\varphi_i : G \rightarrow G_i$, $|G_i| < \infty$ homomorphisms,
 $\bigcap_i \ker \varphi_i = (1)$.

G is [residually finite](#)

Ex.1 A finitely generated group of matrices;

Ex.2 F_m the free group on m free generators x_1, \dots, x_m ;

Ex.3 K/F an infinite Galois extension, $\text{Gal}(K/F)$.



These classes behave differently.

The Burnside Problem (1902)

$$G = \langle a_1, \dots, a_m \rangle, \exists n : \forall g \in G \quad g^n = 1 \stackrel{?}{\implies} |G| < \infty.$$

More generally, what makes a group finite?

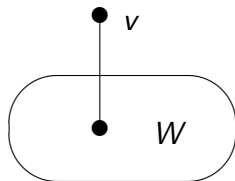
No ([Novikov-Adian, 1968](#))

For Residually Finite Groups: YES ([E.Z., 1991](#))

Expanders.

$\Gamma = (V, E)$ finite connected graph, $\emptyset \neq W \subset V$,

$$\partial W = \{v \in V \mid v \notin W, \text{dist}(v, W) = 1\}$$



$$W \leftrightarrow (W \cup \partial W) \leftrightarrow \dots$$

Definition. $\epsilon > 0$; Γ is an ϵ -expander if $\forall \emptyset \neq W \subset V$ such that $|W| \leq \frac{1}{2}|V|$

$$|W \cup \partial W| \geq (1 + \epsilon)|W|.$$

Expanders

Wanted: infinite family of k -regular graphs $\Gamma_n = (V_n, E_n)$ that are all ϵ -expanders; k, ϵ are fixed, $|V_n| \rightarrow \infty$.

Pinsker, 70s; Barzdin-Kolmogorov, 60s

$G = \langle X \rangle$ finite group

$\text{Cay}(G, X)$ Cayley Graph

$V = G$



if $g_2 = x^{\pm 1}g_1$, $x \in X$.

Connected $2|X|$ -regular graph.

Kazhdan (1967): \exists groups $G = \langle X \rangle$, $|X| < \infty$, with the following property:

Property (T)

$\exists \epsilon > 0 \quad \forall$ unitary representation $\rho : G \rightarrow U(H)$ without $\neq 0$ fixed points: $\forall h \in H \quad \exists x \in X \quad \|xh - h\| \geq \epsilon \|h\|$.

For example, $G = \mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$.

G. Margulis (1981): $G = \langle X \rangle$, $|X| < \infty$, residually finite & has property (T); $\varphi : G \rightarrow G_i$, $|G_i| < \infty$, $X \rightarrow X_i$, $G_i = \langle X_i \rangle$.

Then $\{\mathrm{Cay}(G_i, X_i)\}_i$ is an expander family.

Kassabov, Lubotzky, Breuillard, Green, Tao: any infinite family of finite simple groups \rightsquigarrow expander family.

Growth of Groups.

G a group, $G = \langle x_1, \dots, x_m \rangle$

$$B(n) = \{x_{i_1}^{\pm 1} \cdots x_{i_k}^{\pm 1}, k \leq n\}$$

$$\bigcup_{n \geq 1} B(n) = G.$$

$B(n)$ = ball of radius n with the center at 1 in $\text{Cay}(G, X)$.

$$g(X, n) = |B(n)| < \infty$$

$$g(1) \leq g(2) \leq \dots \quad \text{growth function}$$

Unfortunately, $g(X, n)$ depends on X .

$$\mathbb{N} = \{1, 2, \dots\}$$

$$f, g : \mathbb{N} \rightarrow [1, \infty)$$

Definition. $f \leq g$ asymptotically less than or equal to g if $\exists c \geq 1$:

$$f(n) \leq c g(cn) \quad \text{for all } n.$$

If $f \leq g$, $g \leq f$ then $f \sim g$ asymptotically equivalent.

If $G = \langle X \rangle = \langle Y \rangle$, $|X| < \infty$, $|Y| < \infty$ then

$$g(X, n) \sim g(Y, n).$$

Growth of $G =$ class of equivalence.

J. Milnor (1968):

- (1) is it true that the growth of a group is polynomially bounded iff $G \triangleright H$, $|G : H| < \infty$, H is nilpotent?
- (2) Do there exist groups of intermediate growth?

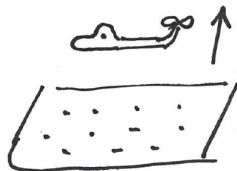
Both problems were solved at about the same time in 1980-1982.

Growth of Groups

Gromov: groups of polynomial growth.

$G \rightsquigarrow \text{Cay}(G)$ metric space

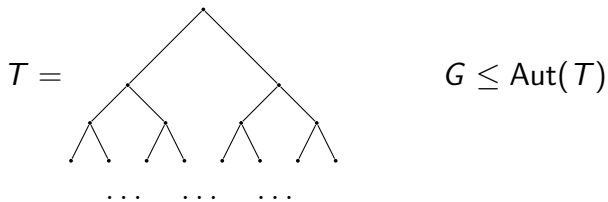
G acts on $\text{Cay}(G)$ by isometries $g : x \rightarrow xg, x \in G$



$\text{Cay}(G) \rightsquigarrow R^n$.

G acts on R^n , $G \rightarrow \text{GL}(n, R)$ linear group.

Grigorchuk: groups of intermediate growth.



Now Branch Groups or Fractal Groups

\rightsquigarrow Dynamical Systems, Number Theory, etc. etc.

Growth of Algebras

Algebras: F a field, $A = F$ -algebra generated by a finite dimensional subspace V .

$$V^n = \sum \underbrace{V \cdots V}_k, \quad k \leq n,$$

$$V^1 \subseteq V^2 \subseteq \dots, \quad \bigcup V^n = A,$$

$$g(V, n) = \dim_F(V^n).$$

If $g(V, n) \leq n^d$ (polynomially bounded) then the minimal such $d =$ Gelfand-Kirillov dimension of A .

Approximate Groups.

G a group, $A \subset G$ a subset (symmetric: $A = A^{-1}$), $k \geq 1$.

The properties:

(1) for $x, y \in A$ $xy^{-1} \in A$ with probability $\geq \frac{1}{k}$;

(2) $|A^2| \leq k|A|$;

(3) A^2 is covered by k right translates of A , $A^2 \subseteq \bigcup_{i=1}^k Ag_i$,

\rightsquigarrow to the same theories.

A is a k -approximate group.

Examples.

(i) $A = \{n \mid -N \leq n \leq N\}$ is a 2-approximate group in \mathbb{Z} .

(ii) d -dimensional arithmetic progression

$$A = \{n_1x_1 + \dots + n_dx_d \mid |n_i| \leq N_i\} \subset \mathbb{Z}$$

is a 2^d -approximate subgroup

- No fast expansion \rightsquigarrow approximate subgroups
- Polynomial growth ([Gromov's Theorem](#))
 \rightsquigarrow balls of radius n are nice approximate subgroups.

Approximate Groups

Freiman-Ruzsa: $A \subseteq \mathbb{Z}$ a k -approximate subgroup

$$\Rightarrow A \subseteq P = \{n_1x_1 + \dots + n_dx_d \mid |n_i| \leq N_i\}, \quad d \leq k, \quad \frac{|P|}{|A|} \leq f(k).$$

Helfgott, Gamburd-Bourgain-Sarnak, Hrushovski, Breuillard-Green-Tao, Pyber-Szabo, ... \Rightarrow a noncommutative version for linear groups.

Side Effects: better understanding (efficient version) of Gromov's theorem, a new approach to Hilbert's 5th Problem.

Profinite and Pro- p Groups.

G residually finite

$$\bigcap \{H \triangleleft G \mid |G : H| < \infty\} = (1).$$

Basis of neighborhoods of 1

The topology is complete = **profinite group** = inverse limit of finite groups.

$$\hat{G} = \text{completion of } G, \quad G \hookrightarrow \hat{G}$$

In any case

$$G \longrightarrow G / \bigcap \{H \triangleleft G \mid |G : H| < \infty\} \longrightarrow \hat{G}$$

Profinite and Pro- p Groups

Example. K/F infinite Galois extension of fields, $\text{Gal}(K/F)$ is profinite.

p a prime number, $\varphi_i : G \rightarrow G_i$, G_i are finite p -groups, $\bigcap_i \ker \varphi_i = (1)$. Then G is residually- p .

Complete topology = pro- p group = inverse limit of finite p -groups.

$G_{\hat{p}}$ pro- p completion, $G \hookrightarrow G_{\hat{p}}$.

In any case

$$G \longrightarrow G / \bigcap \{ H \triangleleft G \mid |G : H| = p^k, k \geq 0 \} \longrightarrow G_{\hat{p}}$$

Profinite and Pro- p Groups

Ex.1 F_m the free group on x_1, \dots, x_m ; $\forall p$ residually- p ,
 $(F_m)_{\hat{p}}$ free pro- p group.

Ex.2 \mathbb{Z}_p p -adic integers, $GL^1(n, \mathbb{Z}_p) = 1_n + pM_n(\mathbb{Z}_p)$ pro- p group.

M. Lazard (1965): $\forall p$ -adic Lie group has an open subgroup that
is $\hookrightarrow GL^1(n, \mathbb{Z}_p)$.

Ex.2' $GL^1(n, R)$, R more general commutative rings.

Polynomial identity: $1 \neq w(x_1, \dots, x_m) \in (F_m)_{\hat{p}}$

$$\forall g_1, \dots, g_m \in G \quad w(g_1, \dots, g_m) = 1.$$

$(p\text{-adic Lie groups}) \subset (\text{PI-groups})$

Possible application

Fontaine-Mazur Conjecture:

$\forall \rho : \text{Gal}(K/\mathbb{Q}) \longrightarrow \text{GL}^1(n, R)$ the image of ρ is finite.