Finite time blowup constructions for supercritical equations

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June 16, 2017

Evolution equations are ordinary differential equations (ODE) or partial differential equations (PDE) that involve a time variable t. Simple examples include

- A first-order ODE $\partial_t u = F(u)$, where $u : [0, T) \to \mathbf{R}^m$ is the unknown field and $F : \mathbf{R}^m \to \mathbf{R}^m$ is the nonlinearity;
- A nonlinear wave (NLW) equation $-\partial_{tt} u + \Delta u = F(u)$, where $u : [0, T) \times \mathbf{R}^d \to \mathbf{R}^m$ is the unknown field and $F : \mathbf{R}^m \to \mathbf{R}^m$ is the nonlinearity;
- A nonlinear Schrödinger (NLS) equation $i\partial_t u + \Delta u = F(u)$, where $u : [0, T) \times \mathbf{R}^d \to \mathbf{C}^m$ is the unknown field and $F : \mathbf{C}^m \to \mathbf{C}^m$ is the nonlinearity.

In this talk we restrict attention to smooth solutions that decay in space.

A physically significant example evolution equation is the (incompressible) Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla) u = \nu \Delta u - \nabla p$$

$$\nabla \cdot u = 0$$

where $u : [0, T) \times \mathbf{R}^3 \to \mathbf{R}^3$ is the unknown velocity field, $p : [0, T) \times \mathbf{R}^3 \to \mathbf{R}$ is the unknown pressure, and $\nu > 0$ is a given constant viscosity.

The Euler equations are the limiting case of the Navier-Stokes equations when $\nu = 0$.

The natural problem to solve in an evolution equation is the initial value problem, when one specifies some initial data at time $t = 0$, and asks if one can construct a solution with this data for later times.

- For the ODE $\partial_t u = F(u)$, one specifies an initial position $u(0) = u_0 \in \mathbf{R}^m$.
- For the NLW $-\partial_{tt} u + \Delta u = F(u)$, one specifies an initial position $u(0) = u_0 : \mathbf{R}^d \to \mathbf{R}^m$ as well as an initial velocity $\partial_t u(0) = u_1 : \mathbf{R}^d \to \mathbf{R}^m$.
- For the NLS $i\partial_t u + \Delta = F(u)$, one specifies an initial position $u(0) = u_0 : \mathbf{R}^d \to \mathbf{C}^m$.

For the Navier-Stokes and Euler equations

$$\partial_t u + (u \cdot \nabla) u = \nu \Delta u - \nabla p$$

$$\nabla \cdot u = 0,$$

one specifies an initial velocity $u_0 : \mathbf{R}^3 \to \mathbf{R}^3$ obeying the incompressibility condition $\nabla \cdot u = 0$. (One does not need to specify an initial pressure, as it can be derived from the initial velocity.)

- Typically, provided that one works with sufficiently regular initial data, it is easy to establish local existence (and uniqueness) for these initial value problems, but global existence is much more problematic.
- For instance, for the ODE initial value problem $\partial_t u = F(u)$, $u(0) = u_0$ with $u_0 \in \mathbf{R}^m$ and $F : \mathbf{R}^m \to \mathbf{R}^m$ given (with F smooth), the Picard existence and uniqueness theorem guarantees the existence of a maximal time of existence $0 < T_* \leq +\infty$ and a unique smooth solution $u : [0, T_*) \to \mathbf{R}^m$ to the initial value problem.
- Furthermore, one has an dichotomy between global existence and finite time blowup: either $T_* = +\infty$, or else $T_* < \infty$ and $u(t) \to \infty$ as $t \to T_*$.

- Even for very simple ODE, such as the Ricatti equation $\partial_t u = u^2$, $u : [0, T_*) \to \mathbf{R}$, one can have finite time blowup, as can be seen for instance with the explicit solution $u(t) = \frac{1}{1-t}$ for $0 \leq t < 1$.
- What is happening with this solution is that when the solution u has size $u \sim 2^n$ (say), $\partial_t u$ is comparable to 2^{2n}, and so u can double in size to $u \sim 2^{n+1}$ in time $t \sim 2^{-n}$. The convergence of the geometric series $\sum_{n=1}^\infty 2^{-n}$ is what makes finite time blowup possible.

- In the ODE case, at least, finite time blowup can be averted if one has a conservation law which is coercive in the sense that it traps the solution in a compact set.
- For instance, consider the Hamiltonian ODE $\partial_{tt} u = -(\nabla_{\mathbf{R}^m} V)(u)$ for $u : [0, T_*) \to \mathbf{R}^m$ and some smooth potential function $V : \mathbf{R}^m \to \mathbf{R}$. This ODE has a conserved energy
$$E(t) := \frac{1}{2}\|\partial_t u(t)\|_{\mathbf{R}^m}^2 + V(u(t))$$
in the sense that $E(t) = E(0)$ for all $t \in [0, T_*)$. If V is *defocusing* in the sense that $V(u) \to +\infty$ as $u \to \infty$, then this conservation law prevents finite time blowup, so we must have global existence.

When one works with a PDE instead of an ODE, is it still the case that a coercive conservation law prevents finite time blowup? The answer is it depends.

An explicit counterexample is given by the one-dimensional focusing quintic NLS

$$i\partial_t u + \frac{1}{2}\partial_{xx} u = -|u|^4 u.$$

This equation has a conserved mass

$$M(t) := \int_{\mathbf{R}} |u(t,x)|^2 \, dx.$$

Nevertheless, there exists a smooth (and even Schwartz) solution that blows up in finite time:

$$u(t,x) := \frac{e^{-i\frac{3}{8(t-1)} + \frac{ix^2}{2(t-1)}}}{(i(t-1))^{1/2}} Q\left(\frac{x}{t-1}\right)$$

where $Q(x) := (3/8)^{1/4} \cosh^{-1/2}(x)$ is the ground state.

Roughly speaking, at times $t \approx 1 - 2^{-n}$, the explicit solution

$$u(t,x) := \frac{e^{-i\frac{3}{8(t-1)} + \frac{ix^2}{2(t-1)}}}{(i(t-1))^{1/2}} Q\left(\frac{x}{t-1}\right)$$

has magnitude $\sim 2^{n/2}$ on an interval of length $\sim 2^n$: the mass

$$M(t) := \int_{\mathbf{R}} |u(t,x)|^2\, dx$$

remains constant, but other function space norms of the solution, such as the Sobolev norm $\|u(t)\|_{H^1(\mathbf{R})}$, go to infinity, leading to finite time blowup (the solution can double in amplitude in time $\sim 2^{-n}$). The mass concentrates to a single point!

- Now let us look at the defocusing three-dimensional NLW

$$-\partial_{tt} u + \Delta u = |u|^{p-1} u$$

for $u : [0, T_*) \times \mathbf{R}^3 \to \mathbf{R}$, where $p > 1$ is a fixed exponent.

- This equation has a coercive conserved quantity, namely the energy

$$E(u(t)) := \int_{\mathbf{R}^3} \frac{1}{2}|\partial_t u(t,x)|^2 + \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{p+1}|u(t,x)|^{p+1}\, dx.$$

- Is this enough to ensure global existence (starting from, say, smooth compactly supported data)?

- Global existence is known for the subcritical case $p < 5$ (Jorgens, 1961) and the critical case $p = 5$ (Grillakis 1990, Shatah-Struwe 1994). Global existence for the supercritical case $p > 5$ remains open.
- A similar situation holds for the defocusing three-dimensional NLS

$$i\partial_t u + \Delta u = |u|^{p-1} u;$$

  despite a similar conserved energy, global existence is only known for the subcritical case $p < 5$ (Ginibre-Velo 1985) and the critical case $p = 5$ (Bourgain 1999, Colliander, Keel, Staffilani, Takaoka, T. 2008).
- What is so special about $p = 5$?

- One can isolate the importance of $p = 5$ using scaling heuristics (or dimensional analysis).
- Suppose at a given time t, the solution $u(t)$ to the NLW

$$-\partial_{tt}u + \Delta u = |u|^{p-1}u$$

oscillates at some frequency $\sim N$ and has an amplitude $\sim A$. Then the dispersive term Δu is expected to have size $\sim N^2 A$, while the nonlinear term $|u|^{p-1}u$ has size $\sim A^p$. Thus one expects the nonlinearity to dominate when $A^p \gg N^2 A$.

- On the other hand, by the uncertainty principle, if u has frequency $\sim N$ then it must spread out, at minimum, on a ball of radius $\sim 1/N$, which has volume $\sim N^{-3}$.
- Thus, the energy

$$E(u(t)) := \int_{\mathbf{R}^3} \frac{1}{2}|\partial_t u(t,x)|^2 + \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{p+1}|u(t,x)|^{p+1}\, dx$$

should be at least $\gtrsim ((NA)^2 + A^{p+1})N^{-3}$. So energy conservation should give a constraint

$$((NA)^2 + A^{p+1})N^{-3} \lesssim 1.$$

- Basic algebra shows that this is incompatible with nonlinear domination $A^p \gg N^2 A$ when $p \leq 5$. (Similarly for NLS instead of NLW.)

- This analysis suggests that finite time blowup might be possible in the energy supercritical case $p > 5$, but only if one can keep all (or most) of the energy concentrated to as small a ball as is consistent with the uncertainty principle as the blowup progresses.
- But can one actually construct a solution that does this?

The answer is yes, if one is willing to work with vector-valued versions of the NLW:

Theorem (T., 2016)

If $p > 5$, then there exists a smooth nonlinearity $V : \mathbf{R}^m \to \mathbf{R}$, homogeneous and positive of degree $p+1$ near infinity, such that the vector-valued NLW $-\partial_{tt} u + \Delta u = (\nabla_{\mathbf{R}^m} V)(u)$ admits smooth compactly supported solutions that blow up in finite time.

A similar result (with some additional technical complications) is true for NLS.

- This blowup result does not directly imply any blowup result for the scalar NLW $-\partial_{tt} u + \Delta u = |u|^{p-1} u$. However, it is a barrier to proving global regularity for such equations, because it shows that any such regularity proof must somehow use a property of the scalar NLW that is not shared by the vector-valued NLW.
- But most of the known proof methods for establishing global regularity apply equally well to both equations.

Some ideas of the proof:

- The NLW $-\partial_{tt}u + \Delta u = (\nabla_{\mathbf{R}^m} V)(u)$ enjoys finite speed of propagation. Because of this, it suffices to construct a blowup solution in a backwards light cone, such as $\{(t,x) : t \leq 0; |x| \leq |t|\}$.
- It is then natural to try to construct a continuously self-similar solution, in which $u(\lambda t, \lambda x) = \lambda^{-\frac{2}{p-1}} u(t,x)$ in the cone for all $\lambda > 0$. But it turns out one can (after using many integration by parts identities) rule out such solutions!

- Instead, we construct a discretely self-similar solution, in which $u(\lambda t, \lambda x) = \lambda^{-\frac{2}{p-1}} u(t,x)$ for all $\lambda \in 2^{\mathbf{Z}}$.
- This compactifies the spacetime domain $\mathcal{M}$ that one is constructing solutions on, to the region $\{(t,x) : -2 \leq t \leq -1 : |x| \leq |t|\}$ with the sides $t = -1$ and $t = -2$ identified.

- To any solution $-\partial_{tt} u + \Delta u = (\nabla_{\mathbf{R}^m} V)(u)$ to NLW, one can associate the stress-energy tensor

$$T_{\alpha\beta} := \langle \partial_\alpha u, \partial_\beta u \rangle - \frac{1}{2}\eta_{\alpha\beta}(\langle \partial^\gamma u, \partial_\gamma u \rangle + V(u))$$

where $\eta_{\alpha\beta}$ is the Minkowski metric (which we use to raise and lower indices). The stress-energy conservation law

$$\partial^\alpha T_{\alpha\beta} = 0$$

encodes all the known conservation laws of NLW (conservation of energy, momentum, angular momentum, etc..)

- To construct the solution u, we first construct the stress-energy tensor $T_{\alpha\beta}$, and then look for u and V that generate this tensor!

- The problem of reconstructing a solution $u : \mathcal{M} \to \mathbf{R}^m$ and potential $V : \mathbf{R}^m \to \mathbf{R}$ given the stress-energy tensor

$$T_{\alpha\beta} := \langle \partial_\alpha u, \partial_\beta u \rangle - \frac{1}{2} \eta_{\alpha\beta} (\langle \partial^\gamma u, \partial_\gamma u \rangle + V(u))$$

  is a more complicated version of the isometric embedding problem: given a smooth symmetric positive definite tensor $g_{\alpha\beta}$ on a compact manifold $\mathcal{M}$, find an injection $u : \mathcal{M} \to \mathbf{R}^m$ such that $g_{\alpha\beta} = \langle \partial_\alpha u, \partial_\beta u \rangle$.
- The latter problem is solved by the Nash embedding theorem. With some elementary calculus (and tools such as the Tietze extension theorem) it is possible to use the Nash embedding theorem to reconstruct u, V from $T_{\alpha\beta}$, assuming that $T_{\alpha\beta}$ obeys some positive definiteness conditions.

- It remains to construct a discretely self-similar stress-energy tensor $T_{\alpha\beta}$ that obeys the conservation law $\partial^\alpha T_{\alpha\beta}$ and some positive definiteness conditions.
- One can simplify matters by making some symmetry reductions on the stress-energy tensor, in particular that it obey rotational and scaling symmetry. (However, if one tries to impose these symmetries on the original solution u, then too many components of the stress-energy vanish to be able to create a finite time blowup.)
- Using the sharp Huygens principle in three dimensions, the problem then boils down to solving some ODE, which is relatively straightforward (but a bit messy).

Now we discuss the Navier-Stokes system

$$\partial_t u + (u \cdot \nabla)u = \nu \Delta u - \nabla p$$

$$\nabla \cdot u = 0.$$

Applying the Leray projection $P := 1 - \Delta^{-1}\nabla(\nabla\cdot)$ to divergence-free vector fields, one can rewrite this system as a nonlinear heat equation

$$\partial_t u = \nu \Delta u + B(u, u)$$

where $B(u, u)$ is the quadratic operator

$$B(u, u) := -P((u \cdot \nabla)u).$$

- Integration by parts reveals the cancellation

$$\langle B(u,u), u \rangle_{L^2(\mathbf{R}^3)} = 0$$

which leads to the energy identity

$$\partial_t \frac{1}{2} \int_{\mathbf{R}^3} |u|^2\, dx = -\nu \int_{\mathbf{R}^3} |\nabla u|^2\, dx.$$

- Unfortunately, in three dimensions the Navier-Stokes equation is supercritical - boundedness of the energy does not seem to preclude finite time blowup!

Indeed, the energy identity is not sufficient by itself:

Theorem (T., 2015)

There exists an "averaged" version $\tilde{B}(u, u)$ of the quadratic operator such that the averaged Navier-Stokes equation $\partial_t u = \nu \Delta u + \tilde{B}(u, u)$ still obeys the energy identity, but admits smooth rapidly decreasing solutions that blow up in finite time.

Previous work of Montgomery-Smith, Chemin-Gallagher-Paicu, Li-Sinai obtained similar results but without energy conservation. The definition of "averaged" is technical and will not be given here.

- Roughly speaking, the idea is to “engineer” the nonlinearity $\tilde{B}(u,u)$ so that when the solution is concentrated at one frequency $\sim 2^n$ and in one ball $B(0,2^{-n})$ at the dual spatial scale, the nonlinearity pushes the energy into the next frequency scale $\sim 2^{n+1}$ and into a smaller ball $B(0,2^{-n+1})$, in a time that decays geometrically with n; this is a low to high frequency cascade.
- The enemy is “Kolmogorov turbulence”: the energy spreading out over many scales (e.g. following a power law). With sufficient spreading, the viscosity term $\nu\Delta u$ can dominate the nonlinear term $\tilde{B}(u,u)$ and create global regularity.
- To prevent thus, one has to “program” the nonlinearity $\tilde{B}(u,u)$ to have some *delay* in it, so that it almost fully transfers energy from one frequency scale $\sim 2^n$ to the next $\sim 2^{n+1}$, before initiating the transfer from $\sim 2^{n+1}$ to $\sim 2^{n+2}$.

- As with the results on NLW (and NLS), this theorem is a barrier to certain approaches to proving global regularity for the Navier-Stokes equations, in that such approaches must somehow use a property of the Navier-Stokes equations that is not true for the averaged Navier-Stokes equation.
- However, there is an important property of Navier-Stokes of this type, namely the vorticity equation

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \Delta \omega$$

obeyed by the vorticity $\omega = \nabla \times u$. The averaged Navier-Stokes equation does not obey this equation.

- In the case of the Euler equations, the vorticity equation has many important consequences. For instance, it implies conservation of helicity $\int_{\mathbf{R}^3} u \times \omega$. It also implies that the vorticity lines (the curves tangent to the vorticity) are transported by the flow.
- Indeed, the Euler equations can be written in vorticity form

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla) u$$

$$u = \nabla \times (-\Delta)^{-1} \omega.$$

One can generalise the Euler equations by considering systems of the form

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla) u$$

$$u = \nabla \times (A\omega)$$

where A is a self-adjoint pseudodifferential operator of the same order as $(-\Delta)^{-1}$. Such equations also enjoy a conserved energy, conserved helicity, transport of vorticity lines, etc..

However, these are not enough to prevent finite time blowup:

Theorem (T., 2016)

There exists a self-adjoint pseudodifferential operator A of the same order as $(-\Delta)^{-1}$ such that the generalised Euler equations

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla) u$$

$$u = \nabla \times (A\omega)$$

admit smooth rapidly decreasing solutions that blow up in finite time.

- The construction is much easier if A is not required to be self-adjoint. We rely on an algebraic trick to embed a non-selfadjoint two-dimensional generalised surface quasi-geostrophic (SQG) equation into a self-adjoint three-dimensional generalised Euler equation via an "axially symmetric with swirl" ansatz.
- The non-selfadjoint SQG blowup is a two-dimensional version of blowup for the one-dimensional non-self-adjoint Burgers-type equation

$$\partial_t \theta(t,x) - \theta(t,2x)\partial_x \theta(t,x) = 0$$

which exhibits blowup from a version of the method of characteristics.

Thanks for listening!