The formula for the real Buchstaber invariants of skeleta of a simplex

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- Definition of Buchstaber invariants $s(K)$.
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Buchstaber invariants

Definition

$K = \{\sigma \subset [m] = \{1, \cdots, m\}\}$: $(n-1)$- dim. simplicial complex.

$$\mathcal{Z}_K = (D^2, S^1)^K \subset (D^2)^m, \quad \mathbb{R}\mathcal{Z}_K = (D^1, S^0)^K \subset (D^1)^m.$$

Definition (Buchstaber invariant)

- Buchstaber invariant $s(K)$ is the maximal dimension of the subtorus of $(S^1)^m$ acting freely on $\mathcal{Z}_K$.
- real Buchstaber invariant $s_{\mathbb{R}}(K)$ is the maximal rank of $(Z_2)^m \cong (S^0)^m$ acting freely on $\mathbb{R}\mathcal{Z}_K$.

- $s(K) \leq m - n$.
- $s(K) \leq s_{\mathbb{R}}(K)$.
- $s(K) = p = s_{\mathbb{R}}(K)$ if $p = 1, 2, 3$.

Buchstaber invariants

Problem (Buchstaber, 2002)

Find an algorithm for calculating the number $s(K)$ based on the combinatorics of K.

- Find an effective criterion for $s(K) = m - n$.
- Behavior of $s(K)$ under some operations and constructions on K.
- Relationship between $s(K)$ and combinatorial invariants of K such as $\gamma(K)$, f-vector, ...
- Find $s(K)$ for special classes of simplicial complex and polytopes.
- Describe the class of simplicial complex and polytopes with given Buchstaber invariants.

Buchstaber invariants

- $s(K) \geq m - \gamma(K) + s(\Delta_{n-1}^{\gamma-1})$, where $\gamma(K)$ is the chromatic number of K.
- $s(K) \leq m - [\log_2(\gamma(K)+1)]$

$$s(\Delta_{n-1}^{m-1}) = ?$$

Theorem (Fukukawa, Masuda,2010)

There exists a set of numbers $\{m_k(b)\}_{k\geq 2,\, b\geq 0}$ such that $s_{\mathbb{R}}(\Delta_{m-p-1}^{m-1}) \geq k$ if and only if $m \leq m_k(p-1)$, where the number $m_k(b)$ is the maximum integer value taken by the linear function $\sum_{v\in\mathbb{Z}_2^k\setminus\{0\}} a_v$ with the constraints

$$\sum_{(u,v)=0} a_v \leq b \textit{ for every } u \in \mathbb{Z}_2^k \setminus \{0\}.$$

Integer linear programming

Theorem (Fukukawa,Masuda,2010)

$s_{\mathbb{R}}(\Delta^{m-1}_{m-p-1}) \geq 2$ *if and only if* $m \leq 3(p-1)$.

Theorem (Fukukawa, Masuda, 2010)

For any $Q \geq 0$ *and* $2^{k-1} - 2^{k-1-\ell} \leq R \leq 2^{k-1} - 2^{k-1-(\ell+1)}$ *for some* $0 \leq \ell \leq k-2$,

$$(2^k-1)Q+R+2^{k-1}-2^{k-1-\ell} \leq m_k((2^{k-1}-1)Q+R) \leq (2^k-1)Q+2R.$$

Conjecture (Fukukawa,Masuda,2010)

$$m_k((2^{k-1}-1)Q+R) = (2^k-1)Q + m_k(R)$$

- If $b \geq (2^{k-1}-1)(2^{k-2}-1)$, then

$$m_k(2^{k-1}-1+b) = 2^k - 1 + m_k(b).$$

Integer linear programming

$(2^k - 1) \times (2^k - 1)$ binary matrix A_k is defined recursively as

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_k = \begin{pmatrix} A_{k-1} & 1 & A_{k-1} \\ 1 \cdots 1 & 0 & 0 \cdots 0 \\ A_{k-1} & 0 & (\mathbf{I} - A_{k-1}) \end{pmatrix}.$$

Problem (primal and dual problem)

Let $k \geq 2$ and $b \geq 0$.

Primal problem	Dual problem
maximize $\sum x_i$	minimize $\sum y_j$
s.t. $A_k \mathbf{x} \leq \mathbf{b} = (b, \cdots, b)^T$,	s.t. $A_k \mathbf{y} \geq \mathbf{b}$,
$\mathbf{x} = (x_1, \cdots, x_{2^k-1}) \geq \mathbf{0}$.	$\mathbf{y} = (y_1, \cdots, y_{2^k-1}) \geq \mathbf{0}$.

Definition

Let $m_k(b)$ be the optimal solution of primal problem.
Let $m_k^*(b)$ be the optimal solution of dual problem.

Solution of dual problem

Lemma

- $m_k^*((2^{k-1}-1)\ell) = (2^k-1)\ell$.
- $m_k^*(a) = m_{k-1}^*(a)$ *for* $0 \le a < 2^{k-2}-1$.
- $m_k^*(a+a') = m_k^*(a) + m_k^*(a')$.

- For convenience, given $k \ge 2$, let

$$\begin{aligned} a &= c_1(2^{k-1}-1) + c_2(2^{k-2}-1) + \cdots c_{k-1}(2^1-1) \\ &:= \overline{c_1, \cdots, c_{k-1}}. \end{aligned}$$

- ex) Let $k=5$. Then
$58 = 3(2^{5-1}-1) + 1(2^3-1) + 2(2^2-1) = \overline{3,1,2,0}$.

Theorem (C)

For $k \ge 2$,

$$m_k^*(\overline{c_1, \cdots, c_{k-1}}) = \overline{c_1, \cdots, c_{k-1}, 0}.$$

Solution of primal problem

- Let $0 \leq b < 2^{k-1} - 1$. For optimal solution $\mathbf{y}$ of dual problem $A_k\mathbf{y} \geq \mathbf{b}$, it cannot be guaranteed that $\mathbf{y} \leq (1, \cdots, 1)^T$.

Definition

Let $k \geq 2$ and $0 \leq b \leq 2^{k-1} - 1$. We define

$$d_k(b) := \min\{\max\{y_i\} \mid \mathbf{y} : \text{optimal solution of } A_k\mathbf{y} \geq \mathbf{b} \text{ and } \mathbf{y} \geq \mathbf{0}\} - 1.$$

- If a is the smallest number such that there exists an optimal solution vector $\mathbf{y}^*$ of the dual problem $A_k\mathbf{y} \geq \mathbf{b}$ satisfying

$$\mathbf{y}^* \leq (a, \cdots, a)^T,$$

then $d_k(b) = a - 1$ and vice versa.

Solution of primal problem

Theorem (C)

For $a, b > 0$ *with* $a + b = 2^{k-1} - 1$ *and* $\ell \geq d_k(a)$,

$$m_k((2^{k-1} - 1)\ell + b) + m_k^*(a) = (2^k - 1)(\ell + 1).$$

Moreover, if

$$(2^{k-1} - 1) - b = \overline{c_2, \cdots, c_{k-1}},$$

for $\ell \geq d_k(\overline{c_2, \cdots, c_{k-1}}) = d_k(a)$,

$$m_k((2^{k-1} - 1)(\ell + 1) - \overline{c_2, \cdots, c_{k-1}}) = (2^k - 1)(\ell + 1) - \overline{c_2, \cdots, c_{k-1}, 0}.$$

Theorem (C)

Let $k \geq 2$*. For* $p-1 = (2^{k-1}-1)(\ell+1) - \overline{c_2, \cdots, c_{k-1}}$ *and* $\ell \geq d_k(\overline{c_2, \cdots, c_{k-1}})$*,*

$$s_{\mathbb{R}}(\Delta^{m-1}_{m-p-1}) \geq k \Leftrightarrow \frac{m + \overline{c_2, \cdots, c_{k-1}, 0}}{(p-1) + \overline{c_2, \cdots, c_{k-1}}} \leq \frac{2^k - 1}{2^{k-1} - 1}$$

$$\Leftrightarrow m \leq 2(p-1) + (\ell+1) - \sum c_i.$$

- This is a generalization of $s_{\mathbb{R}}(\Delta^{m-1}_{m-p-1}) \geq 2 \Leftrightarrow m \leq 3(p-1)$.

Preperiodicity

- $m_k(b)$ is preperiodic for large value of b. i.e., $m_k(b)$(viewed as a function of b) is periodic with the period $T = (2^{k-1} - 1)$ and the shift $S = (2^k - 1)$.

$$m_k(b+T) = m_k(b) + S.$$

- The moment at which the behavior of $m_k(b)$ becomes predictable and preperiodic depends on the remainder of b modulo period.
- This moment is encoded by the constant $d_k(b)$. i.e.,

$$\left\lfloor \frac{b}{2^{k-1}-1} \right\rfloor \geq d_k(2^{k-1} - 1 - a),$$

where a is the remainder of b divided by $(2^{k-1} - 1)$.
- $\ell \geq d_k(2^{k-1} - 1 - a)$ is just the moment at which $m_k(b)$ is preperiodic.

Preperiodicity

- The meaning of preperiodicity of $s_{\mathbb{R}}(\Delta^{m-1}_{m-p-1})$ is described as follows: Let $p = (2^{k-1} - 1)\ell_1 + p_1 = (2^k - 1)\ell_2 + p_2$. Suppose that

$$s_{\mathbb{R}}(\Delta^{m-1}_{m-p-1}) = k \text{ if and only if } A < m \leq B.$$

Then

$$s_{\mathbb{R}}(\Delta^{m'-1}_{m'-(p+2^{k-1}-1)-1}) = k \text{ if and only if } A + 2^k < m' \leq B + (2^k - 1),$$

provided $\ell_1 \geq d_k(2^{k-1} - 1 - p_1)$ and $\ell_2 \geq d_{k+1}(2^k - 1 - p_2)$.

Preperiodicity

- Note that $21 = 15 + 6$ and $d_5(9) = d_6(10) = 0$.

Example

Let $k = 5$ and $m = 40$. Then

$$s_{\mathbb{R}}(\Delta^{40-1}_{40-p-1}) = 5 \Leftrightarrow p = 21.$$

Preperiodicity of $s_{\mathbb{R}}(\Delta^{m-1}_{m-p-1})$ means

$$s_{\mathbb{R}}(\Delta^{71-1}_{71-p-1}) = 5 \Leftrightarrow p = 36.$$

Property of $d_k(b)$

Theorem (C)

Let $k \geq 2$ and $b > 0$. Then the following holds:

- $d_k(b) \begin{cases} = 0 & \text{if } k \leq 4 \\ = 0 \text{ or } 1 & \text{if } 5 \leq k \leq 7 \\ \leq k - 6 & \text{if } k \geq 7 \end{cases}$
- $d_k(b) \leq \begin{cases} \frac{k-3}{2} & \text{if } k \text{ is odd} \\ \frac{k-2}{2} & \text{if } k \text{ is even} \end{cases}$
- $d_k(b) = 0$ *for* $2^{k-1} - k \leq b \leq 2^{k-1} - 1$.
- *If* $b = \overline{c_2, \cdots, c_{k-1}}$ *satisfies*

$$c_2 \cdot (k-1) + c_3 \cdot (k-2) + \cdots + c_{k-1} \cdot 2 \leq nk,$$

then $d_k(b) = n - 1$.

Thank you