Algebra topology of configuration space

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1. configuration space $F(X, n)$
2. orbit configuration space $F_G(X, n)$
3. orbit configuration space over small covers and toric manifolds
4. future work

configuration space $F(X,n)$

Definition: ordered configuration space

X be a space. $F(X,n) = \{(x_1, ..., x_n) \in X^n | x_i \neq x_j, \forall i \neq j\}$

group action $\Sigma_n \curvearrowright F(X,n)$

Definition:unordered configuration space

$C(X,n) = F(X,n)/\Sigma_n$

$B(X,n) \triangleq \pi_1(C(X,n))$
$\alpha : B(X,n) \to \Sigma_n$
pure braid group $P(X,n) \triangleq ker(\alpha) = \pi_1(F(X,n))$

1962 Fadell, Neuwirth calculate $\pi_i(F(M, n))$
let M be a manifold and $dim(M) \geq 2$.
$Q_m = \{q_1, ..., q_m\} \subseteq M$ discrete.
$F_{m,n}(M) = \{(p_1, ..., p_n) | p_i \in M - Q_m, p_i \neq p_j, i \neq j\} \subseteq F(M, n)$

Theorem

$\pi : F_{m,n}(M) \to M - Q_m$ and n>1 is a locally trivial fiber space with fiber $F_{m+1,n-1}$. if $m \geq 1$, π admits a cross section. when m=0, there may not exsit a cross section.

Theorem

*For any manifold M, $\pi_i(F_{1,n-1}) = \bigoplus_{k=1}^{n-1} \pi_i(M - Q_k), i \geq 2$
if $\pi : F_{0,n} \to M$ admits a cross section,
then $\pi_i(F(M, n)) = \bigoplus_{k=0}^{n-1} \pi_i(M - Q_k), i \geq 2$*

Example

$$\pi_i(F(\mathbb{R}^r, n)) = \bigoplus_{k=1}^{n-1} \pi_i(\underbrace{S^{r-1} \vee \ldots \vee S^{r-1}}_{k}), i \geq 2$$

Example

for $M = S^r$, r odd

$$\pi_i(F(S^r, n)) = \pi_i(S^r) \bigoplus \bigoplus_{k=1}^{n-2} \pi_i(\underbrace{S^{r-1} \vee \ldots \vee S^{r-1}}_{k}), i \geq 2$$

Remark

If M is differentiable and admits a non-vanishing vector field, then π admits a cross section.
If M is an even dimensional sphere, π admits a cross section only when n=2.

1995 F.R.Cohen calculate the cohomology ring of $F(\mathbb{R}^m, n)$

Theorem

If $n \geq 2, H^*(F(\mathbb{R}^m, n), \mathbb{Z}) \cong < A_{ij} | 1 \leq i < j \leq n, deg A_{ij} = m-1, A_{ij}^2 = 0, A_{ij}A_{ik} = A_{kj}A_{ik} - A_{kj}A_{ij} >$

$\pi_{ij} : F(\mathbb{R}^m, n) \to F(\mathbb{R}^m, 2)$ $A_{ij} = \pi_{ij}^*([a])$
$[a]$ is generator of $H^{m-1}(F(\mathbb{R}^m, 2))$

1997, Ziegler compute $X = S^r$ case

other works

1. Fulton and MacPherson determined the rational homotopy type of configuration spaces of non-singular compact complex algebraic varieties.
2. 1993, Totaro determined the rational cohomology ring of configuration spaces of smooth complex projective varieties.
3. Bodigheimer compute the homology groups of $C(X, n)$ of odd dimensional manifolds X.
4. Loffler and Milgram $\mathbb{Z}_2$ homology groups of $C(X, n)$ for any manifold X.

orbit configuration space

group action $G \times X \to X$

$G_x = \{g \in G | gx = x\}$ isotropy subgroup of x

1. trivial action : $G_x = G$
2. free action :$G_x = \{e\}$
3. semi-free action: $G_x = G$ or $\{e\}$
4. general action: such as $\mathbb{Z}_2^n \curvearrowright M^n$, $T^n \curvearrowright M^{2n}$

Definition:orbit configuration space

$$F_G(M^n, k) = \{(x_1, ..., x_k) \in (M^n)^k | Gx_i \cap Gx_j = \phi, \forall i \neq j\}$$

1. trivial action: $F_G(M^n, k) = F(M^n, k)$
2. free action: $\pi : M \to M/G$ is a bundle projection.

   then there is a principal G^k-bundle $F_G(M, k) \to F(M/G, k)$

2000 Feichner,Ziegler determined explicitly $H^*(F_{\mathbb{Z}_2}(S^k, n), \mathbb{Z})$

orbit configuration space over small covers and quasi-toric manifolds

$\pi_d : G_d^m \curvearrowright M^{dm} \to P^m$ locally standard.
when d=1, $G_d^m = \mathbb{Z}_2^m$, M^m is a small cover
when d=2, $G_d^m = T^m$, M^{2m} is a quasi-toric manifold

Theorem (2010, Junda Chen, Zhi Lu and Jie Wu)

let $\pi_d : M \to P$ be a dm-dimensional G_d^m-manifold over a simple convex polytope P where d=1,2. then the Euler characteristic of $F_{G_d^m}(M, n)$ is

$$\chi(F_{G_d^m}(M,n)) = \begin{cases} (-1)^{mn} \sum\limits_{I=(n_1,\ldots,n_s)} \prod_{k=1}^{s} \mathcal{C}_I h_P(1-2^{n_k}), d=1 \\ \chi(F(M,n)) = \sum\limits_{I=(n_1,\ldots,n_s)} \mathcal{C}_I h_P(1)^s, d=2 \end{cases}$$

where $I = (n_1, \ldots, n_s)$ runs over all partitions of n,
$\mathcal{C}_I = \frac{n!(-1)^{n-s}}{r_1!r_2!\ldots r_k!n_1n_2\ldots n_s}$, r_k is the time number that n_k appears in I.

Theorem (2010, Junda Chen, Zhi Lu and Jie Wu)

Let $\pi_d : M \to P$ be a dm-dimensional G_d^m-manifold over a simple convex polytope P where d=1,2.Then there is an equivariant strong deformation retraction of $(F_{G_d^m}(M, 2)$ onto

$$X_d(M,2) = \bigcup_{F_1, F_2 \in \mathcal{F}(P), F_1 \cap F_2 = \phi} (\pi_d^{-1})^{\times 2}(F_1 \times F_2)$$

where $\mathcal{F}(P)$ is the set of all faces of P.

consider the local representation $G_d^m \curvearrowright \mathbb{R}^{dm}$
for d=1 $F_{\mathbb{Z}_2^m}(\mathbb{R}^m, n)$ can be regarded as the complement of subspace arrangement

Let $\mathcal{A}$ be a complex hyperplane arrangement.
$\mathcal{A} = \{H_1, ..., H_n\} \subseteq \mathbb{C}^{m+1}$
Orlik Solomon Algebra
$E_1 = \bigoplus_{H \in \mathcal{A}} Ke_H, E = \wedge(E_1), deg(e_H) = 2m$
$I = I(\mathcal{A})$ be the ideal of E generated by ∂e_S,
for all dependent S,$S \subseteq \mathcal{A}$. S is dependent if $r(\cap S) < |S|$

Let M(A) be the complement space,then $H^*(M(\mathcal{A})) \cong E(\mathcal{A})/I(\mathcal{A})$

(Arnold's presentation)
let $G = \{g_H | H \in \mathcal{A}\}$,then $\pi_1(M) =< G | \bigcup_{p \in P} R_p >$
G generators,R_p relators.

(MacPherson-Goresky) compute the homology group of complement of affine subspace arrangements.

(Ziegler's result)
cf:*the cohomology rings of complements of subspace arrangements*

Let $\mathcal{A}$ be a (≥ 2)-arrangement in W with geometric intersection lattice L. Fix an arbitrary linear order on the set of atoms of L. Then the integral cohomology ring of the complement $M(\mathcal{A})$ has the presentation

$$0 \to I \to T^*(\bigoplus_{\sigma ind.} \mathbb{Z}e_\sigma) \xrightarrow{\pi} H^*(M(\mathcal{A}); \mathbb{Z}) \to 0$$

where T^* denotes the tensor algebra and the sum in the middle is over all independent sets σ of atoms of L,and $\pi(e_\sigma) \in H^{d(W)-d(\cap\sigma)-|\sigma|}(M(\mathcal{A}); \mathbb{Z})$.The ideal I of relations is generated by the following three families of elements.

1. For every minimal dependent set $\sigma = \{a_0, ..., a_k\}$ atoms of L:

$$\sum_{i=0}^{k}(-1)^i e_{\sigma-\{a_i\}}$$

.

2. For all pairs σ, τ of independent sets of atoms of L such that $d(\bigcap\sigma) + d(\bigcap\tau) - d(\bigcap\sigma \cap \bigcap\tau) = d(W)$:

$$e_\sigma \wedge e_\tau - \epsilon_{\cap\sigma,\cap\tau}(-1)^{|\tau|(d(W)-d(\cap\sigma))}(-1)^{sign(\sigma,\tau)}e_{\sigma\cup\tau}$$

where $sign(\sigma, \tau)$ is the sign of the permutation that orders the elements of σ followed by the elements of τ, each already ordered, ascendingly according to the chosen linear order.

3. For all pairs σ, τ of independent sets atoms of L such that $d(\bigcap\sigma) + d(\bigcap\tau) - d(\bigcap\sigma \cap \bigcap\tau) \neq d(W)$:

$$e_\sigma \wedge e_\tau$$

go back to local representation $F_{\mathbb{Z}_2^m}(\mathbb{R}^m, n)$

check the proof of above theorem, we can find that Ziegler's result can be applied to compute its integer cohomology ring

1. determine$\pi_1(F_{\mathbb{Z}_2^2}(\mathbb{R}^2, n))$
   if $m > 2, \pi_1(F_{\mathbb{Z}_2^m}(\mathbb{R}^m, n)) = 0$
2. determine algebra topology of $F_{\mathbb{Z}_2^m}(M^m, n)$
3. if it is possible, I want to calculate particular examples of orbit configuration space over toric manifolds

looking forward to your suggestions!

Thanks for listening!