Symplectic Capacities
from Hamiltonian Circle Actions

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jointly with
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1. A motivating example and definitions

2. Some known results and statement of the main results

3. Examples

4. Proof of the main theorems
(M, \omega) : a symplectic manifold of \text{dim}_\mathbb{R} M = 2n, 
\text{i.e. a real } 2n\text{-dim manifold with a closed non-degenerate 2-form } \omega.
A motivating Example

\((M, \omega)\) : a symplectic manifold of \(\dim_{\mathbb{R}} M = 2n\), i.e. a real \(2n\)-dim manifold with a closed non-degenerate 2-form \(\omega\).

\(B^{2n}(r) := \{x \in \mathbb{R}^{2n} \mid |x| < r\} \subset (\mathbb{R}^{2n}, \omega_{\text{std}})\),

where \(\omega_{\text{std}} = dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n}\).
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Question

For what \(r > 0\) does there exists a symplectic embedding

\((B^{2n}(r), \omega_{\text{std}}) \hookrightarrow (M^{2n}, \omega)\)?
Consider $S^2$ with the symplectic form $\sigma$ s.t.

$$\int_{S^2} \sigma = \pi$$

Let $M := S^2 \times S^2$ with $\omega := \sigma \times \sigma = \pi_1^*\sigma + \pi_2^*\sigma$, where $\pi_i : S^2 \times S^2 \to S^2$ is the projection to the $i$-th component. Then

$$\text{vol}(M) \overset{\text{def}}{=} \int_M \omega^2 / 2 = \left( \int_{S^2} \sigma \right)^2 = \pi^2.$$
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There exists a symplectic embedding $(B^2(1 - \epsilon/2), \omega_{\text{std}}) \hookrightarrow (S^2, \sigma)$. We thus have the symplectic embedding

$$B^4(1 - \epsilon) \hookrightarrow B^2(1 - \epsilon/2) \times B^2(1 - \epsilon/2) \hookrightarrow M.$$ 

Note that $\lim_{\epsilon \to 0} \text{vol}(B^4(1 - \epsilon)) = \pi^2 / 2$. 
Proposition

No larger ball can be symplectically embedded in $M = S^2 \times S^2$, i.e., $\nexists B^4 (1 + \varepsilon) \hookrightarrow M :$ symplectic embedding for any $\varepsilon > 0$. 

Proof.

This proposition follows from the following theorem of Gromov.

Theorem (Gromov)

Let $B^2_n (r) \hookrightarrow (M^2_n, \omega)$ be a symplectic embedding.

If $GW_{M^A, k} ([pt], \alpha_2, \ldots, \alpha_k) \neq 0$ for $0 \neq A \in H_2 (M; \mathbb{Z})$, then $\pi r^2 \leq \omega (A)$.

For $M = S^2 \times S^2$, choose $A = [S^2 \times pt] \in H_2 (M; \mathbb{Z})$.

Since $GW_{M^A, 1} = 1 \neq 0$ and $\omega (A) = \int_{S^2 \times pt} \omega = \pi$, we have $\pi r^2 \leq \pi$, hence $r \leq 1$. 

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For $M = S^2 \times S^2$, choose $A = [S^2 \times pt] \in H_2(M : \mathbb{Z})$. Since $GW^M_{A,1} = 1 \neq 0$ and $\omega(A) = \int_{S^2 \times pt} \omega = \pi$, we have $\pi r^2 \leq \pi$, hence $r \leq 1$. 

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The Gromov width of a $2n$-dim symplectic manifold $(M^{2n}, \omega)$ is

$$w_G(M) := \sup\{\pi r^2 \mid (B^{2n}(r), \omega_{\text{std}}) \xrightarrow{\text{symp.emb}} (M^{2n}, \omega)\}.$$
Definition of Gromov width

**Definition**

The Gromov width of a $2n$-dim symplectic manifold $(M^{2n}, \omega)$ is

$$w_G(M) := \sup \{ \pi r^2 \mid (B^{2n}(r), \omega_{\text{std}}) \xrightarrow{\text{symp.emb}} (M^{2n}, \omega) \}.$$ 

- In the previous example of $M = S^2 \times S^2$, we get $w_G(M) = \pi$: the lower bound is given from the construction of embeddings, the upper bound is given by the Gromov theorem.

- Gromov width is one of the symplectic capacities for symplectic manifolds, and we will discuss another symplectic capacity, called Hofer-Zehnder capacity later.
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Some known results

Theorem (Gromov’s non squeezing theorem)

\[ \exists B^{2n}(r) \xrightarrow{\text{symp. emb.}} B^2(R) \times \mathbb{R}^{2n-2} \iff r \leq R. \]

Hence \( w_G(B^2(R) \times \mathbb{R}^{2n-2}) = \pi R^2. \)
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Theorem (Karshon-Tolman, Lu)

\( M = Gr(k, m) : \text{ Grassmann manifold of } k\text{-planes in } \mathbb{C}^m \)

\( \omega : U(m)\text{-invariant symplectic form s.t.} \)

\[ [\omega] = c_1(M) = m \cdot \text{gen} \in H^2(M : \mathbb{Z}) \cong \mathbb{Z} \]

then \( w_G(M) = m. \)
An almost complex structure on a $2n$-dim manifold $M$ is a tangent bundle automorphism $J : TM \to TM$ such that $J^2 = -\text{Id}$.

$J$ is compatible with a symp form $\omega$ if $\omega(\cdot, J\cdot)$ defines a Riem metric.
Statement of main results

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Let $(S^2, j)$ denote the holomorphic 2-sphere.

A **$J$-holomorphic sphere** in $(M, J)$ is a smooth map $u : (S^2, j) \to (M, J)$ such that $du \circ j = J \circ du$. 
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Definition

$(M, J)$ is Fano if any non-constant $J$-holomorphic sphere $u : (S^2, j) \to (M, J)$ has positive the Chern number, i.e., $c_1(M)(u_*[S^2]) > 0$. 
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$(M, J)$ is Fano if any non-constant $J$-holomorphic sphere $u : (S^2, j) \to (M, J)$ has positive the Chern number, i.e., $c_1(M)(u_*[S^2]) > 0$.

A symplectic manifold $(M, \omega)$ with a symplectic action of a group $G$ is Fano if $\exists$ $\omega$-compatible $G$-invariant almost complex structure $J$ s.t. $(M, J)$ is Fano.
An $S^1$ action on $(M, \omega)$ is Hamiltonian if $\omega$ is $S^1$-invariant and

$$\iota_X \omega = -dH$$

for some smooth function $H : M \to \mathbb{R}^1$, where $X$ is the fundamental vector field of the $S^1$-action. $H$ is called a moment map of the action.
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**Some Notation**

$H_{\text{max}} := \text{maximal critical value of } H$.

$F_{\text{max}} := H^{-1}(H_{\text{max}})$.

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Theorem (1) $(M, \omega)$: closed semi-free $S^1$ Hamiltonian symplectic Fano manifold. If $F_{\text{max}} = \{\text{pt}\}$, then $w_G(M) = H_{\text{max}} - H_{\text{smax}}$. 

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**Theorem (1)**

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If $F_{\text{max}} = \{pt\}$, then

$$w_G(M) = H_{\text{max}} - H_{\text{smax}}.$$
Hofer-Zehnder capacity

For a smooth function $K : M \to \mathbb{R}^1$ its Hamiltonian flow $\Phi_K$ is the flow generated by the vector field $X_K$ defined by $dK = -\iota_{X_K} \omega$.

$K$ is admissible if

1. $K^{-1}(K_{\text{max}})$ and $K^{-1}(K_{\text{min}})$ contain open subsets $^1$,
2. $\Phi_K$ has no non-constant periodic orbit of period $< 1$.

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The **Hofer-Zehnder capacity** of $M$ is defined to be

$$c_{HZ}(M) := \sup\{K_{\text{max}} - K_{\text{min}} \mid K \text{ is admissible}\}.$$
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Theorem (2)

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A motivating example and definitions

Some known results and statement of the main results

Examples

Proof of the main theorems
Example (1. Karshon-Tolman)

\( M = Gr(k, m) \): complex Grassmann manifold; may assume \( k \leq m - k \).

\( \omega = U(n) \)-invariant symplectic form s.t. \( [\omega] = c_1(M) = m \cdot \text{gen} \in H^2(M) \cong \mathbb{Z} \).
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Consider $S^1$ action on $\mathbb{C}^m$ defined by

$$t \cdot (z_1, \ldots, z_m) = (tz_1, \ldots, tz_k, z_{k+1}, \ldots, z_m).$$

It induces a semi-free Hamiltonian $S^1$ action on a Fano (in fact, monotone) manifold $M$. 
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Decompose

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  \mathbb{C}^m = \mathbb{C}^k \times 0 \oplus 0 \times \mathbb{C}^{m-k} =: V_1 \oplus V_2.
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Let $W \subset \mathbb{C}^m$ be an $S^1$-invariant subspace corr. to a fixed pt $[W] \in M^{S^1}$.

Any $v = v_1 + v_2 \in W$ where $v_i \in V_i \cap W$ for $i = 1, 2$.

So $W$ is the product of subspaces of $V_1$ and $V_2$. 
Thus any $S^1$-fixed point component of $M$ is of the form

$$Gr(k_1, k) \times Gr(k_2, m - k), \quad k_1 + k_2 = k.$$
Example (1. Karshon-Tolman continued)

Thus any $S^1$-fixed point component of $M$ is of the form

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Decompose the tangent space $T_W Gr(k, m)$ as

$$\text{Hom}(W, W^\perp) = \text{Hom}(W \cap V_1 \oplus W \cap V_2, W^\perp \cap V_1 \oplus W^\perp \cap V_2)$$
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where the $S^1$ weights of the first and fourth terms equal to 0, the second term equals to $-1$, and the third term equals to 1.
Example (1. Karshon-Tolman continued)

By a result of McDuff-Tolman we know for each fixed point component $F$ of $S^1$ action on $M$, its moment map image is

$$H(F) = \sum S^1 \text{ weights at } F.$$
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Thus

$$H_{\text{max}} = k(m - k) \quad \text{when} \quad k_1 = k$$
$$H_{\text{smax}} = k(m - k) - m \quad \text{when} \quad k_1 = k - 1$$
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Therefore $w_G(M) = H_{\text{max}} - H_{\text{smax}} = m$, and $c_{HZ}(M) = km$. 
Example (2. Product of manifolds)

\((M_i, \omega_i)\): semi-simple Hamilton \(S^1\) Fano symp. mfd for \(i = 1, \ldots n\).

\(H_i : M_i \to \mathbb{R}\): moment map for \((M_i, \omega_i)\).

Assume \(H_i^{-1}(H_{i\text{max}}) = \{\text{pt}\}\).
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Assume \(H_i^{-1}(H_{i_{\text{max}}}) = \{\text{pt}\}\).

Let \(M := \prod M_i\), and \(\omega = \omega_1 \wedge \cdots \wedge \omega_n\).

\(\Rightarrow (M, \omega)\) is a semi-simple Hamilton \(S^1\) Fano symp. mfd with the diagonal \(S^1\) action, whose moment map is \(H = \sum H_i : M \to \mathbb{R}\).
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Then we have

\[
\begin{align*}
    w_g(M) &= \min\{w_G(M_i)\} \\
    c_{HZ}(M) &= \sum c_{HZ}(M_i)
\end{align*}
\]
Example (3. Assumption "semi-free" is needed)

$M := \mathbb{C}P^2 \# \mathbb{C}P^2$: blow-up of toric $\mathbb{C}P^2$ at a fixed point.

We may choose the size of the blow-up so that $M$ is monotone.
Example (3. Assumption "semi-free" is needed)

\[ M := \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} : \text{blow-up of toric } \mathbb{CP}^2 \text{ at a fixed point.} \]

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**Figure:** Moment map image of \( M = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \).

- \( \phi_1 : (0, 1) \)
- \( \phi_2 : (-1, -2) \)
**Example (3. Assumption "semi-free" is needed)**

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**Figure**: Moment map image of \( M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \).

\[ T^2 \]-weights at a fixed pt = two outward prim. vec. from the corr. vertex.
For a rel. prime \((a, b)\) define a sub circle action induced by the inclusion

\[
S^1 \hookrightarrow S^1 \times S^1, \quad t \mapsto (t^a, t^b).
\]

Then \(S^1\) weights at a fixed point \(= (T^2\text{-weights}) \cdot (a, b)\).
Example (3. continued)

For a rel. prime \((a, b)\) define a sub circle action induced by the inclusion

\[ S^1 \hookrightarrow S^1 \times S^1, \quad t \mapsto (t^a, t^b). \]

Then \(S^1\) weights at a fixed point = \((T^2\)-weights) \cdot \((a, b)\).

We consider the following two sub circle actions:

- \(\phi_1\): circle action given by \((a, b) = (0, 1)\) with moment map \(H_1\),
- \(\phi_2\): circle action given by \((a, b) = (-1, -2)\) with moment map \(H_2\).
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For \(\phi_1\)-action (Let \(P_{ij} := D_i \cap D_j\).)

\[ F_{\text{max}} = P_{23} \text{ with } S^1\text{-weights } (1, -1) \cdot (0, 1) = -1, (0, -1) \cdot (0, 1) = -1 \]

\[ \Rightarrow H_1(F_{\text{max}}) = H_{1_{\text{max}}} = 2 \]

\[ F_{\text{max}} = P_{34} \text{ with } S^1\text{-weights } (0, 1) \cdot (0, 1) = 1, (1, -1) \cdot (0, 1) = -1 \]

\[ \Rightarrow H_1(F_{\text{max}}) = H_{1_{\text{max}}} = 0 \]

\[ F_{\text{min}} = D_1 \text{ with } S^1\text{-weights } (1, -1) \cdot (0, 1) = -1, (1, 0) \cdot (0, 1) = 0 \]

\[ \Rightarrow H_1(F_{\text{min}}) = H_{1_{\text{min}}} = -1 \]
Example (3. continued)

For a rel. prime \((a, b)\) define a sub circle action induced by the inclusion

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\[
F_{\text{min}} = D_1 \text{ with } S^1\text{-weights } (1, -1) \cdot (0, 1) = -1, (1, 0) \cdot (0, 1) = 0
\]

\(\Rightarrow H_1(F_{\text{min}}) = H_{1_{\text{min}}} = -1\)

Hence \(w_G(M) = H_{1_{\text{max}}} - H_{1_{\text{smax}}} = 2 - 0 = 2\), and

\[
C_{HZ} = H_{1_{\text{max}}} - H_{1_{\text{min}}} = 2 + 1 = 3.
\]
Example (3. continued)

For $\phi_2$-action

$F_{\text{max}} = P_{14}$ with $S^1$-weights $(-1, 1) \cdot (-1, -2) = -1$, 
$(0, -1) \cdot (-1, -2) = -1$

$\Rightarrow H_2(F_{\text{max}}) = H_{2_{\text{max}}} = 2$

$F_{\text{smax}} = P_{34}$ with $S^1$-weights $(0, 1) \cdot (-1, -2) = -2$, $(1, -1) \cdot (-1, -2) = 1$

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Hence $H_{2_{\text{max}}} - H_{2_{\text{max}}} = 2 - 1 = 1 \neq w_G(M)$, and $H_{2_{\text{max}}} - H_{2_{\text{min}}} = 2 + 3 = 5 \neq C_{HZ}$.

Note that $\phi_2$-action is not semi-free.
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Hence $H_{2\text{max}} - H_{2\text{smax}} = 2 - 1 = 1 \neq w_G(M)$, and
\[
H_{2\text{max}} - H_{2\text{min}} = 2 + 3 = 5 \neq C_{HZ}.
\]
Note that $\phi_2$-action is not semi-free.

Note that the Gromov width and the Hofer-Zehnder capacity are independent of group actions.
A motivating example and definitions

Some known results and statement of the main results

Examples

Proof of the main theorems
Proof of Theorem (1)

Lower Bound
Exactly the same proof as in [Karshon-Tolman] works in our case to prove that $H^{-1}(a, b]$ is symplectomorphic to the ball $B^{2n}(b - a)$ where $a = H_{\text{smax}}$ and $b = H_{\text{max}}$. Hence we have

$$w_G(M) \geq H_{\text{max}} - H_{\text{smax}}.$$
Proof of Theorem (1)

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Exactly the same proof as in [Karshon-Tolman] works in our case to prove that $H^{-1}(a, b]$ is symplectomorphic to the ball $B^{2n}(b - a)$ where $a = H_{s\text{max}}$ and $b = H_{\text{max}}$. Hence we have

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**Upper Bound**
By Theorem (Gromov) it is enough to show that

$$GW_{A,k}^M([\text{pt}], \alpha_2, \ldots, \alpha_k) \neq 0$$

for some $A \in H_2(M : \mathbb{Z})$, and $\alpha_i \in H^*(M : \mathbb{Q})$. 
Proof of Theorem (1)

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Upper Bound
By Theorem (Gromov) it is enough to show that

$$GW_{A,k}^M([pt], \alpha_2, \ldots, \alpha_k) \neq 0$$

for some $A \in H_2(M : \mathbb{Z})$, and $\alpha_i \in H^*(M : \mathbb{Q})$. So we will prove that $GW_{A,3}^M([pt], \alpha_2, \alpha_3) \neq 0$ for some $A \in H_2(M : \mathbb{Z})$, and $\alpha_i \in H^*(M : \mathbb{Q})$. 

We only consider the case when the genus of surface $g = 0$. 

$(M, \omega, J)$: $2n$-dimensional symplectic manifold with $\omega$-compatible almost complex structure $J$.

$(S^2, j)$: 2-sphere $S^2 = \mathbb{C}P^1$ with the complex structure $j$.

Fix a homology class $A \in H_2(M; \mathbb{Z})$.

Let $\alpha_1, \ldots, \alpha_k \in H^*(M; \mathbb{Q})$.

**Definition**

The Gromov-Witten invariant $GW_M^A(K)(\alpha_1, \ldots, \alpha_k)$ is the "signed number" $a$ of $J$-holomorphic spheres $u: S^2 \to M$ whose image intersects with the cycles representing the homology classes $PD(\alpha_1), \ldots, PD(\alpha_k)$.

If $\sum_k i_k = 1$ and $\deg \alpha_i = 2n + c_1(A) + 2k - 6$, and 0 otherwise.

It is defined as an intersection number of certain pseudocycles, and the number is rational. Sometimes we will confuse cohomology classes with homology classes for simplicity.
Gromov-Witten invariant

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\(^b\)sometimes we will confuse cohomology classes with homology classes
Consider the following commutative ring

\[ \Lambda^{\text{univ}} := \left\{ \sum_{k \in \mathbb{R}} r_k t^k \mid r_k \in \mathbb{Q}, \forall c \in \mathbb{R} \# \{k < c \mid r_k \neq 0\} < \infty \right\}. \]
Quantum Cohomology

Consider the following commutative ring

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Now let \( \Lambda := \Lambda^{univ}[q, q^{-1}] \), a graded ring with \( \text{deg} \ q = 2 \).
Quantum Cohomology

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**Definition (Small Quantum Cohomology)**

Quantum cohomology with coefficient ring \( \Lambda \) is

\[ QH^*(M : \Lambda) := (H^*(M : \mathbb{Z})/\text{torsion}) \otimes_{\mathbb{Z}} \Lambda \]

with the following multiplication:
Quantum Cohomology

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Definition (Small Quantum Cohomology)

Quantum cohomology with coefficient ring \( \Lambda \) is

\[ QH^*(M : \Lambda) := \left( H^*(M : \mathbb{Z})/\text{torsion} \right) \otimes_{\mathbb{Z}} \Lambda \]

with the following multiplication: for \( a, b \in H^*(M : \mathbb{Z}) \)

\[ a \ast b := \sum_{A \in H_2(M;\mathbb{Z})} (a \ast b)_A \otimes q^{c_1(A)} t^{\omega(A)}, \]

where \( (a \ast b)_A \in H^{|a|+|b|-2c_1(A)}(M) \) is the unique class satisfying \( \int_M (a \ast b)_A \cup c = GW^M_{A,3}(a, b, c) \) for any \( c \in H^*(M : \mathbb{Z}) \).
Let $\phi : S^1 \times M \to M$ be a Hamiltonian $S^1$-action on $(M^{2n}, \omega)$. 

$P_\phi$ is determined by the homotopy class $[\phi] \in \pi_1(\text{Ham}(M, \omega))$, and has a symplectic structure extending the fiberwise symplectic form.

$H_{\sec^2}(P_\phi) := \{ \sigma \in H^2(P_\phi) | \pi^*(\sigma) = [S^2] \}$. 

$i^* : H^*(M) \to H^*(P_\phi)$: the push-forward map. 

c_{vert} := 1$-st Chern class of vertical tangent bundle of $P_\phi$. 

$u_\phi \in H^2(P_\phi)$: the unique class s.t. $u_\phi|_M = [\omega]$, $u_{n+1} = 0$. 

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Seidel element

Let $\phi : S^1 \times M \to M$ be a Hamiltonian $S^1$-action on $(M^{2n}, \omega)$. Consider the $S^1$-action on $S^3 \times M$ defined by

$$ t \cdot (z_1, z_2, x) = (tz_1, tz_2, \phi(t, x)). $$
Let $\phi : S^1 \times M \to M$ be a Hamiltonian $S^1$-action on $(M^{2n}, \omega)$. Consider the $S^1$-action on $S^3 \times M$ defined by

$$t \cdot (z_1, z_2, x) = (tz_1, tz_2, \phi(t, x)).$$

Let $P_\phi := S^3 \times S^1 M$, and consider the following fiber bundle with fiber $M$

$$M \xrightarrow{i} P_\phi \xrightarrow{\pi} S^2.$$
Let $\phi : S^1 \times M \to M$ be a Hamiltonian $S^1$-action on $(M^{2n}, \omega)$. Consider the $S^1$-action on $S^3 \times M$ defined by

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$P_\phi$ is determined by the homotopy class $[\phi] \in \pi_1(\text{Ham}(M, \omega))$, and has a symplectic structure extending the fiberwise symplectic form. $\text{Ham}(M, \omega)$ : Hamiltonian symplectomorphism group of $(M, \omega)$. 
Let $\phi : S^1 \times M \rightarrow M$ be a Hamiltonian $S^1$-action on $(M^{2n}, \omega)$. Consider the $S^1$-action on $S^3 \times M$ defined by

$$t \cdot (z_1, z_2, x) = (tz_1, tz_2, \phi(t, x)).$$

Let $P_\phi := S^3 \times_{S^1} M$, and consider the following fiber bundle with fiber $M$

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$P_\phi$ is determined by the homotopy class $[\phi] \in \pi_1(\text{Ham}(M, \omega))$, and has a symplectic structure extending the fiberwise symplectic form.

$\text{Ham}(M, \omega) :$ Hamiltonian symplectomorphism group of $(M, \omega)$.

Let us fix some notation.

$H^2_{\text{sec}}(P_\phi) := \{ \sigma \in H_2(P_\phi) \mid \pi_*(\sigma) = [S^2] \}$.

$i_* : H^*(M : \mathbb{Q}) \rightarrow H^{*+2}(P_\phi : \mathbb{Q}) :$ the push-forward map.

$c_{\text{vert}} := 1$-st Chern class of vertical tangent bundle of $P_\phi$.

$u_\phi \in H^2(P_\phi : \mathbb{R}) :$ the unique class s.t. $u_\phi|_M = [\omega], u^{n+1}_\phi = 0.$
The **Seidel element** of the action $\phi$ is defined to be

$$S(\phi) := \sum_{\sigma \in H^2_{\text{sec}}(P_{\phi})} b_\sigma \otimes q^{c_{\text{vert}}(\sigma)} t^{u_{\phi}(\sigma)} \in QH^0(M : \Lambda),$$

where $b_\sigma \in H^*(M : \mathbb{Q})$ is the unique class satisfying

$$\int_M b_\sigma \cup c = GW^{P_{\phi}}_{\sigma,1}(i_\ast c)$$

for any $c \in H^*(M : \mathbb{Q})$. Theorem (McDuff)

Theorem $S : \pi_1(\text{Ham}(M, \omega)) \to QH^0(M : \Lambda) \times [\phi] \mapsto S(\phi)$ is a group homomorphism.
Definition

The **Seidel element** of the action $\phi$ is defined to be

$$S(\phi) := \sum_{\sigma \in H^{sec}(P_\phi)} b_\sigma \otimes q_{\text{vert}}^c t_{\mu_{\phi}}(\sigma) \in QH^0(M : \Lambda),$$

where $b_\sigma \in H^*(M : \mathbb{Q})$ is the unique class satisfying

$$\int_M b_\sigma \cup c = GW^{P_\phi}_{\sigma,1}(i_*c)$$

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Theorem (McDuff)

$$S : \pi_1(\text{Ham}(M, \omega)) \to QH^0(M : \Lambda)^\times \quad [\phi] \mapsto S(\phi)$$

is a group homomorphism.
A computational result

The following is shown in [McDuff-Tolman].

∀ fixed component $F \in \pi_0(M^{S^1})$ with index $\alpha$ and $\forall c_F \in H^i(F : \mathbb{Q})$,

$\exists c_F^+ \in H^{i+\alpha}(M : \mathbb{Q})$ called upward extension and

$\exists c_F^- \in H^{i+2n-\alpha-\dim F}(M : \mathbb{Q})$ called downward extension s.t.

if $B_F = \{c_F\}$: basis of $H^*(F : \mathbb{Q})$ and $B_F^+ := \{c_F^+\}$, then

$B^+ := \bigcup_{F \in \pi_0(M^{S^1})} B_F^+$ is a basis of $H^*(M : \mathbb{Q})$, and, similarly for $B^-$. 
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$\mathcal{B}^+ := \bigcup_{F \in \pi_0(M^{S_1})} \mathcal{B}_F^+$ is a basis of $H^*(M : \mathbb{Q})$, and, similarly for $\mathcal{B}^-$. 

Geometric interpretation

Assume $S^1$-action is semi-free.

Choose a Riem. metric $g$ assoc. to a generic $S^1$-invariant $\omega$-compatible al. cpx. struct. $J$ on $M$. For a generic submanifold $C \subset F$,

$W^u(C) := \{ x \in M \mid \lim_{t \to -\infty} \psi_t(x) \in C \}$, called unstable mfd of $C$

where $\psi_t = -(\text{gradient flow of the moment map } H)$.
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$W^u(C) := \{ x \in M \mid \lim_{t \to -\infty} \psi_t(x) \in C \}$, called unstable mfd of $C$
where $\psi_t = -(\text{gradient flow of the moment map } H)$. Then the cohomology class represented by $W^u(C)$
$= \text{downward extension } [C]^-$. 
$[C]^+$ is defined by stable manifold of $C$. 

Let $F \in \pi_0(M^{S^1})$, and choose $x \in F$.
Let $\sigma_F := [S^3 \times_{\phi} \{x\}] \in H^2_{\text{sec}}(P_\phi : \mathbb{Z})$.
(Note that this class does not depend on the choice of $x$.)
Let $F \in \pi_0(M^{S^1})$, and choose $x \in F$. Let $\sigma_F := [S^3 \times \phi \{x\}] \in H^\text{sec}_2(P_\phi : \mathbb{Z})$. (Note that this class does not depend on the choice of $x$.)

$\forall \sigma \in H^\text{sec}_2(P_\phi : \mathbb{Z})$ can be written as the $\sigma_F + B$ for some $B \in H_2(M : \mathbb{Z})$. We let

$$a_B := b_{\sigma_F + B}.$$
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**Theorem (McDuff-Tolman)**

Let $\phi$ be a semi-free Hamiltonian $S^1$-action on a closed Fano symplectic manifold $M$. For $c_F \in H^*(F : \mathbb{Q})$

$$S(\phi) * c_F^- = c_F^+ \otimes q^{m(F)} t^{-H(F)} + \sum_{\substack{B \in H_2(M; \mathbb{Z}) \\ \omega(B) > 0}} a_{B,c_F} \otimes q^{m(F) + c_1(B)} t^{-H(F) + \omega(B)},$$

where $m(F) = \sum$ weights at $F$, and $a_{B,c_F} \in H^*(M : \mathbb{Q})$ is s.t.

$$\int a_{B,c_F} \cup c = GW_{a_B}^{P_\phi} (i_*c_F^-, i_*c) \quad \text{for any } c \in H^*(M : \mathbb{Q}).$$
On the other hand, the following theorem shows that, in our situation, Seidel element consists of a single term.

**Theorem (3)**

If \( \phi \) is a semi-free Hamiltonian \( S^1 \) action on a closed Fano symplectic manifold \((M, \omega)\), then the Seidel element of \( \phi \) is

\[
S(\phi) = [F_{\text{max}}] \otimes q^{m(F_{\text{max}})} t^{-H_{\text{max}}},
\]

where \( F_{\text{max}} = H^{-1}(H_{\text{max}}) \), and \( m(F_{\text{max}}) = \sum \text{weights at } F_{\text{max}} \).

Using Theorem (3) and the previous Theorem (McDuff-Tolman) we could prove Theorem (1) as follows.
Proof of Theorem (1)

Theorem (1)

\((M, \omega)\): closed semi-free \(S^1\) Hamiltonian symplectic Fano manifold. If \(F_{\text{max}} = \{ \text{pt} \}\), then

\[ w_G(M) = H_{\text{max}} - H_{\text{smax}}. \]
Proof of Theorem (1)

Theorem (1)

\((M, \omega)\): closed semi-free \(S^1\) Hamiltonian symp. Fano manifold.

If \(F_{\text{max}} = \{pt\}\), then

\[ w_G(M) = H_{\text{max}} - H_{\text{smax}}. \]

Proof of Theorem (1)

Choose a fixed component \(F\) such that \(H(F) = H_{\text{smax}}\).

From Theorem (3) and Theorem (McDuff-Tolman) we have

\[
\left( [F_{\text{max}}] \otimes q^{m(F_{\text{max}})} t^{-H_{\text{max}}} \right) \ast c_F^- = c_F^+ \otimes q^{m(F)} t^{-H(F)} + \sum_{B \in H_2(M; \mathbb{Z})} \omega(B) > 0
\]
Proof of Theorem (1) continued

\[ \text{LHS} = \sum_{B \in H_2(M;\mathbb{Z})} \left( [F_{\text{max}}] * c_F \right)_B q^{m(F_{\text{max}})} + c_1(B) t^{H_{\text{max}} + \omega(B)} \]

\[ = \sum_{B \in H_2(M;\mathbb{Z})} \left( [F_{\text{max}}] * c_F \right)_B q^{m(F)} t^{-H_{\text{sm}} + \omega(B)} + \text{other terms} \]

where 
\[ ( [F_{\text{max}}] * c_F - F_{\text{max}} ) \] is the unique class in \( H^*(M;Q) \) s.t.
\[ \int ( [F_{\text{max}}] * c_F - F_{\text{max}} ) \cup c = GW_{M}B,3([F_{\text{max}}],c,F_{\text{max}},c) \] for any \( C \in H^*(M;Q) \).
Proof of Theorem (1) continued

LHS = \[ \sum_{B \in H_2(M;\mathbb{Z})} ([F_{\text{max}}] * c_F^-)_B \ q^{m(F_{\text{max}})+c_1(B)} t^{-H_{\text{max}}+\omega(B)} \]

= \[ \sum_{B \in H_2(M;\mathbb{Z})} ([F_{\text{max}}] * c_F^-)_B \ q^{m(F)} t^{-H_{\text{smax}}} + \text{other terms} \]

where \( ([F_{\text{max}}] * c_F^-)_B \) is the unique class in \( H^*(M : \mathbb{Q}) \) s.t.

\[ \int ([F_{\text{max}}] * c_F^-)_B \cup c = GW^{M}_{B,3}([F_{\text{max}}], c_F^-, c) \]

for any \( C \in H^*(M : \mathbb{Q}) \).
Since $c_F \neq 0$, by (1) $\exists A \in H_2(M : \mathbb{Z})$ s.t. $\omega(A) = H_{\text{max}} - H_{\text{smax}}$ and $([F_{\text{max}}] * c_F)_A \neq 0$. 
Since \( c_F \neq 0 \), by (1) \( \exists A \in H_2(M : \mathbb{Z}) \) s.t. \( \omega(A) = H_{\text{max}} - H_{\text{smax}} \) and 
\[
([F_{\text{max}}] \ast c_F^-)_{A} \neq 0.
\]
Now choose \( \alpha \in H^*(M : \mathbb{Q}) \) s.t. 
\[
\int_M ([F_{\text{max}}] \ast c_F^-)_A \cup \alpha \neq 0.
\]
Then
\[
GW_{A,3}(\text{pt}, c_F^-, \alpha) = \int_M ([F_{\text{max}}] \ast c_F^-)_A \cup \alpha \neq 0.
\]
This proves Theorem (1) \( \square \)
Proof of Theorem (2)

Theorem (2)

\((M, \omega)\): closed semi-free \(S^1\) Hamiltonian symplectic Fano manifold.

If \(F_{\text{max}} = \{\text{pt}\}\), then

\[ c_{HZ}(M) = H_{\text{max}} - H_{\text{min}}. \]
Proof of Theorem (2)

**Theorem (2)**

\((M, \omega)\): closed semi-free \(S^1\) Hamiltonian symp. Fano manifold.

If \(F_{\text{max}} = \{\text{pt}\}\), then

\[
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\]

**Proof of Theorem (2)**

**Lower bound**

Moment map \(H\) is not admissible.

But \(\forall \epsilon > 0\) and for some open nbds \(F_{\text{max}} \subset U\) and \(F_{\text{min}} \subset V\), we can modify \(H\) to a function \(K\) s.t.

1. \(K\) is admissible
2. \(K_{\text{max}} = H_{\text{max}} - \pi \epsilon^2\) and \(K_{\text{min}} = H_{\text{min}} + \pi \epsilon^2\)
3. \(K = H\) on \(M - (U \cup V)\).

Then \(c_{HZ}(M) \geq H_{\text{max}} - H_{\text{min}}\).
Proof of Theorem (2)-continued

Upper bound
We apply Theorem (McDiff-Tolman) for $F = F_{\min}$ and choose $c_F = [\text{pt}]$. Then $c_F^- = [\text{pt}]$. 

By comparing the coefficients of $q^m(F) t - H_{\min}$, similar to (1), we have

$$c + F = \sum_{b \in H_2(M: \mathbb{Z})} c_1(B) = m(F) - m(F_{\max})w(B) = H_{\max} - H_{\min}(\lfloor F_{\max} \rfloor \ast c - F).$$

As in the proof of Theorem (1), \exists A \in H_2(M: \mathbb{Z}) and \alpha \in H^\ast(M: \mathbb{Q}) s.t. \omega(A) = H_{\max} - H_{\min}, and $GW_{MA}, 3([\text{pt}], [\text{pt}], \alpha, ..., \alpha) \neq 0$. Now the theorem follows from the following result of Lu:

Theorem (Lu)
If $GW_{MA}, k([\text{pt}], [\text{pt}], \alpha, ..., \alpha) \neq 0$ for $k \geq 2$, then $c_{HZ}(M) \leq \omega(A)$. 

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Proof of Theorem (2)-continued

Upper bound

We apply Theorem (McDiff-Tolman) for $F = F_{\text{min}}$ and choose $c_F = [pt]$. Then $c_F^- = [pt]$.

By comparing the coefficients of $q^{m(F)} t^{-H_{\text{min}}}$, similar to (1), we have

$$c_F^+ = \sum_{b \in H_2(M;\mathbb{Z})} (([F_{\text{max}}] * c_F^-)_B$$

$$c_1(B) = m(F) - m(F_{\text{max}})$$

$$w(B) = H_{\text{max}} - H_{\text{min}}$$
Proof of Theorem (2)-continued

Upper bound
We apply Theorem (McDiff-Tolman) for $F = F_{\text{min}}$ and choose $c_F = [\text{pt}]$. Then $c_F^- = [\text{pt}]$.
By comparing the coefficients of $q^{m(F)} t^{-H_{\text{min}}}$, similar to (1), we have

$$c_F^+ = \sum_{b \in H_2(M;\mathbb{Z})} ([F_{\text{max}}] * c_F^-)_B$$

As in the proof of Theorem (1), $\exists A \in H_2(M;\mathbb{Z})$ and $\alpha \in H^*(M;\mathbb{Q})$ s.t. $\omega(A) = H_{\text{max}} - H_{\text{min}}$, and $GW^M_{A,3}([\text{pt}],[\text{pt}],\alpha) \neq 0$. Now the theorem follows from the following result of Lu:

**Theorem (Lu)**

If $GW^M_{A,k}([\text{pt}],[\text{pt}],\alpha_3,\ldots,\alpha_k) \neq 0$ for $k \geq 2$, then $c_{HZ}(M) \leq \omega(A)$. 
Proof of Theorem (3)

**Theorem (3)**

If $\phi$ is a semi-free Hamiltonian $S^1$ action on a closed Fano symplectic manifold $(M, \omega)$, then the Seidel element of $\phi$ is

$$S(\phi) = [F_{\text{max}}] \otimes q^{m(F_{\text{max}})} t^{-H_{\text{max}}},$$

where $F_{\text{max}} = H^{-1}(H_{\text{max}})$, and $m(F_{\text{max}}) = \sum \text{weights at } F_{\text{max}}$. 
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Proof of Theorem (3)

By the work of McDuff-Tolman, we have the following expression of the Seidel element:

$$S(\phi) = [F_{\text{max}}] \otimes q^{m(F_{\text{max}})} t^{-H_{\text{max}}} + \sum_{B \in H_2(M; \mathbb{Z}) \atop \omega(B) > 0} a_B \otimes q^{m(F_{\text{max}}) + c_1(B)} t^{-H_{\text{max}} + \omega(B)}.$$
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To show that $a_B = 0$ it is enough to show that $\int_M a_B \cup c_F^- = 0$ for all $c_F \in H^*(F : \mathbb{Q})$ and for all fixed components $F$. 
Proof of Theorem (3)-continued

From the property of downward expansion

\[
\deg c_F^- = \deg c_F + \dim M - \text{index}(F) - \dim F
= (\dim M - \text{index}(F)) - (\dim F - \deg c_F)
\leq \dim M - \text{index}(F)
\leq \dim M + 2m(F) \quad \text{(because semi-free)}
\]  (2)
Proof of Theorem (3)-continued

From the property of downward expansion

\[
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\leq \dim M + 2m(F) \quad \text{(because semi-free)} \quad (2)
\]

Case 1: \( F = F_{\text{max}} \)

From the Fano condition, i.e., \( c_1(B) \geq 0 \),

\[
\deg a_B = -2m(F_{\text{max}}) - 2c_1(B) < -2m(F_{\text{max}}) \quad (3)
\]
Proof of Theorem (3)-continued

From the property of downward expansion

\[ \deg c_F^- = \deg c_F + \dim M - \text{index}(F) - \dim F \]
\[ = (\dim M - \text{index}(F)) - (\dim F - \deg c_F) \]
\[ \leq \dim M - \text{index}(F) \]
\[ \leq \dim M + 2m(F) \quad \text{(because semi-free)} \quad (2) \]

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From the Fano condition, i.e., \( c_1(B) \geq 0 \),

\[ \deg a_B = -2m(F_{\text{max}}) - 2c_1(B) < -2m(F_{\text{max}}) \quad (3) \]

From (2) and (3), we have

\[ \deg a_B + \deg c_{F_{\text{max}}}^- < \dim M, \quad \text{hence} \quad \int_M a_B \cup c_{F_{\text{max}}}^- = 0. \]
Case 2: $F \neq F_{\text{max}}$

Assume $\int_M a_B \cup c_F \neq 0$ for some $c_F$. We may assume that $c_F$ is represented by a generic smooth sub manifold $C \in F$. 
Case 2: $F \neq F_{\text{max}}$

Assume $\int_M a_B \cup c_F \neq 0$ for some $c_F$. We may assume that $c_F$ is represented by a generic smooth sub manifold $C \in F$. By a result of [McDuff-Tolman], there exists $S^1$-invariant $J$-holom. stable map in the class $B$ that intersect both $F_{\text{max}}$ and $\overline{W^u(C)}$. Then there is a chain of spheres joining two fixed points $x \in F_{\text{max}}$ and $y \in \overline{W^u(C)}$, and we can write $B = \sum_{i} B_i + \sum_{j} C_j$ where $B_i$ is the homology class of sphere in the chain, and $C_j$ is the class of a sphere not in the chain.
Case 2: $F \neq F_{\text{max}}$

Assume $\int_M a_B \cup c_F \neq 0$ for some $c_F$. We may assume that $c_F$ is represented by a generic smooth sub manifold $C \in F$.

By a result of [McDuff-Tolman], $\exists$ $S^1$-invariant $J$-holom. stable map in the class $B$ that intersect both $F_{\text{max}}$ and $W^{u}(C)$.

This stable map is a connected union of $S^1$-invariant $J$-holom. spheres $u_i : S^2 \to M$, and the images are either contained in a fixed components or gradient spheres joining two fixed points.
Case 2: \( F \neq F_{\text{max}} \)

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By a result of [McDuff-Tolman], \( \exists S^1\)-invariant \( J \)-holom. stable map in the class \( B \) that intersect both \( F_{\text{max}} \) and \( W^u(C) \).

This stable map is a connected union of \( S^1\)-invariant \( J \)-holom. spheres \( u_i: S^2 \to M \), and the images are either contained in a fixed components or gradient spheres joining two fixed points.

Then there is a chain of spheres joining two fixed points \( x \in F_{\text{max}} \) and \( y \in W^u(C) \), and we can write \( B = \sum k_i B_i + \sum l_i C_j \) where \( B_i \) is the homology class of sphere is the chain, and \( C_j \) is the class of a sphere not in the chain.
Then from the Fano condition we have the inequality

\[ c_1(B) \geq \sum c_1(B_i) \geq m(y) - m(x) \geq m(F) - m(F_{\text{max}}). \]

\(^2\)Here the second inequality follows from a result by McDuff-Tolman showing that for the gradient sphere \( B_i \) connecting two fixed points from \( y_i \) to \( x_i \) we have \( c_1(B_i) = m(y_i) - m(x_i) \).
Proof of Theorem (3)-continued

Then from the Fano condition we have the inequality

\[ c_1(B) \geq \sum c_1(B_i) \geq m(y) - m(x) \geq m(F) - m(F_{\text{max}}). \]

This implies

\[ \deg a_B = -2m(F_{\text{max}}) - 2c_1(B) \leq 2m(F_{\text{max}}) - 2m(F) + 2m(F_{\text{max}}) = -2m(F). \]

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Since \( F \neq F_{\text{max}} \) the inequality in (2) is strict, i.e.,

\[ \deg c_F^- < \dim M + 2m(F). \]

Therefore \( \int_M a_b \cup c_F^- = 0 \) by the dimension reason.

---

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Theorem (Gromov)

Let $\phi : B^{2n}(r) \hookrightarrow (M^{2n}, \omega)$ be a symplectic embedding. If $GW_{A,k}^M([pt], \alpha_2, \ldots, \alpha_k) \neq 0$ for $0 \neq A \in H_2(M : \mathbb{Z})$, then

$$\pi r^2 \leq \omega(A).$$
Idea of the proof of Theorem (Gromov)

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Idea of Proof

Let $J$ be a $\omega$-compatible al. cpx structure on $M$. Since $GW^M_{A,k}([pt], \alpha_2, \ldots, \alpha_k) \neq 0$,

$$\exists u : (S^2, i) \to (M, J) : J\text{-holomorphic sphere in the class } A \text{ intersecting a point } P \in M. \text{ We may assume that } \phi(0) = P.$$
Idea of the proof of Theorem (Gromov)

**Theorem (Gromov)**

Let \( \phi : B^{2n}(r) \hookrightarrow (M^{2n}, \omega) \) be a symplectic embedding. If \( GW^M_{A,k}([pt], \alpha_2, \ldots, \alpha_k) \neq 0 \) for \( 0 \neq A \in H_2(M : \mathbb{Z}) \), then

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**Idea of Proof**

Let \( J \) be a \( \omega \)-compatible al. cpx structure on \( M \).

Since \( GW^M_{A,k}([pt], \alpha_2, \ldots, \alpha_k) \neq 0, \)

\[ \exists u : (S^2, i) \to (M, J) : J \text{-holomorphic sphere in the class } A \text{ intersecting a point } P \in M. \]

We may assume that \( \phi(0) = P \).

Let \( X := S^2 \cap u^{-1}(\phi(B^{2n}(r))) \subset S^2 \), and consider the composition

\[ \phi^{-1} \circ u : X \to B^{2n}(r). \]

This composition is a \( J_{\text{std}} \)-holomophic curve, which in local coordinates, looks like a holomorphic map \( f : \mathbb{C} \to \mathbb{C}^n \).
The image $S$ of a holomorphic map $f : \mathbb{C} \to \mathbb{C}^n$ is a minimal surface. If $0 \in S$, then by the monocity formula

$$\text{Area}(S \cap B^{2n}(r)) \geq \pi r^2.$$
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$$\text{Area}(S \cap B^{2n}(r)) \geq \pi r^2.$$ 

Let $S := \phi^{-1} \circ u(X) \subset B^{2n}(r)$ be the image of the composition. Then

$$\omega(A) \geq \text{Area}(S) \geq \pi r^2,$$

which proves the theorem. $\square$
Thank you!