The cohomology ring of the symmetric square of quaternionic projective space

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References

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2. Minoru Nakaoka. *Cohomology theory of a complex with a transformation of prime period and its applications*. Journal of the Institute of Polytechnics, Osaka City University, Volume 7, Number 1-2, Series A (1956), 51-102.

Preliminaries

Symmetric squares $SP^2(\mathbb{HP}^n)$ of quaternionic projective space

The symmetric square of $\mathbb{HP}^n$ is the quotient space

$$SP^2(\mathbb{HP}^n) = \mathbb{HP}^n \times \mathbb{HP}^n / \sim$$

by identifying $(z, v) \sim (v, z)$ where $z, v \in \mathbb{HP}^n$, that is

$$SP^2(\mathbb{HP}^n) = \mathbb{HP}^n \times \mathbb{HP}^n / \mathbb{Z}/2.$$

Decomposition of $SP^2(\mathbb{HP}^n)$

$SP^2 = L \cup_A N$

The symmetric square $SP^2(\mathbb{HP}^n)$ can be decomposed as

$$SP^2(\mathbb{HP}^n) = L_n \cup_{A_n} N_n$$

where $A_n = L_n \cap N_n$ and N_n is a closed neighbourhood of the diagonal $\triangle := \{(z, z) \mid z \in \mathbb{HP}^n\} \subset SP^2(\mathbb{HP}^n)$, also L_n is the closure of the complement $SP^2(\mathbb{HP}^n) \setminus N_n$.

A_n and L_n

More about A_n, L_n and N_n

- $N_n \simeq \triangle \cong \mathbb{HP}^n$
- $L_n \simeq$ space of all unordered pairs of distinct points on $\mathbb{HP}^n$
- A_n may be described as the total space of the (real)projectivisation $\mathbb{RP}(\tau_{\mathbb{HP}^n})$ of the tangent bundle of $\mathbb{HP}^n$,

$$\mathbb{RP}^{4n-1} \longrightarrow A_n \longrightarrow \mathbb{HP}^n.$$

Main results

$H^*(SP^2;\mathbb{Z}) := H^*(SP^2(\mathbb{HP}^\infty);\mathbb{Z})$

Theorem

The integral cohomology ring of $SP^2(\mathbb{HP}^\infty)$ can be given by the following.
Let $s,\ell,i \geq 1, \quad m \geq 0, \quad 0 < j < 2i,$ then

$$H^*(SP^2;\mathbb{Z}) \cong \mathbb{Z}\Big[\,(1/2)^{s-1}h^s, \quad (1/2)^m g^\ell h^m, \quad t_{i,j}\,\Big] \,/\, \mathrm{I}$$

where

$$\mathrm{I} = \Big(\,2t_{i,j}, \quad t_{i,j}\cdot t_{k,l}, \quad t_{i,j}\cdot(1/2)^{s-1}h^s, \quad t_{i,j}\cdot(1/2)^m g^\ell h^m\Big)$$

and

$$|g| = 4, \quad |h| = 8, \quad |t_{i,j}| = 4i+2j+1.$$

Main results (continue)

$H^*(SP^2_n;\mathbb{Z}) := H^*(SP^2(\mathbb{HP}^n);\mathbb{Z})$

Proposition

The homomorphism η^*, induced by inclusion $\mathbb{HP}^n \subset \mathbb{HP}^\infty$

$$\eta^* : H^*(SP^2;\mathbb{Z}) \to H^*(SP^2_n;\mathbb{Z})$$

is surjective; that is,

the integral cohomology ring of $SP^2(\mathbb{HP}^n)$ can be described as

$$H^*(SP^2_n;\mathbb{Z}) \cong H^*(SP^2;\mathbb{Z})/\ker\eta^*$$

We will explain $\ker\eta^*$ later.

Main methods

From the decomposition of the symmetric square

$$SP^2(\mathbb{HP}^n) = L_n \cup_{A_n} N_n,$$

and the fact $SP^2_n/N_n \cong L_n/A_n$ we obtain:

1. the commutative ladder of cofibre sequences

$$\begin{array}{ccccccc}
N_n & \longrightarrow & SP^2_n & \xrightarrow{u} & SP^2_n/N_n & \longrightarrow & \cdots \\
\uparrow & & \uparrow f & & r \uparrow \cong & & \\
A_n & \longrightarrow & L_n & \longrightarrow & L_n/A_n & \longrightarrow & \cdots
\end{array}$$

2. take $n \to \infty$
3. apply $H^*(-)$ to get a map of long exact sequences(LES).

$H^*(A) := H^*(A_{n\to\infty})$ and $H^*(L) := H^*(L_{n\to\infty})$

How to compute $H^*(A)$ and $H^*(L)$?

The following are useful,

- $A \simeq \mathbb{RP}^\infty \times \mathbb{HP}^\infty$,
- $L \simeq S^\infty \times_{\mathbb{Z}/2} (\mathbb{HP}^\infty \times \mathbb{HP}^\infty) =: B$,

then

$$H^*(B;\mathbb{Z}) \cong \mathbb{Z}[x,y,w]/(2w,xw), \quad |w| = 2,\ |x| = 4,\ |y| = 8$$

may be deduced from the fibration

$$\mathbb{RP}^4 \to B \to Gr_{\mathbb{H}}(\infty, 2).$$

Note: Alternatively we can use Fred Roush's (unpublished)results for $H^*(B;\mathbb{Z})$.

Definitions of x, y and w

Recall
$H^*(Gr_{\mathbb{H}}(\infty,2);\mathbb{Z}) \cong \mathbb{Z}[p_1,p_2]$, where $|p_1| = 4$, $|p_2| = 8$,
$H^*(\mathbb{RP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[a]/(2a)$, where $|a| = 2$.

Definition

Given the maps

$$i : B \to Gr_{\mathbb{H}}(\infty,2), \quad \pi : B \to \mathbb{RP}^\infty,$$

generators $w, x, y \in H^*(B;\mathbb{Z})$ are defined by

$$w := \pi^*(a), \quad x := i^*(p_1) + w^2, \quad y := i^*(p_2) + w^4.$$

Definitions of $g, h, \cdots$

Definition

Given $f : L \to SP^2$ from the ladder, and $B := L$,

$$f^* : H^*(SP^2; \mathbb{Z}) \to H^*(B; \mathbb{Z});$$

then the generators $(1/2)^{s-1}h^s,\ (1/2)^m g^\ell h^m \in H^*(SP^2; \mathbb{Z})$ are defined uniquely by

$$f^* : (1/2)^{s-1}h^s \longmapsto 2y^s$$

$$f^* : (1/2)^m g^\ell h^m \longmapsto x^\ell y^m$$

where $\ell \geq 1,\ m \geq 0$.

Definition of $t_{i,j}$

Let $* = 4i + 2j + 1, \quad i \geq 1,$ and $0 < j < 2i,$ then the generator $t_{i,j} \in H^*(SP^2;\mathbb{Z})$ is defined to be

$$t_{i,j} := u^* \circ (r^*)^{-1}\ \delta(z^i w^j),$$

where $\delta(z^i w^j) \in H^*(L/A;\mathbb{Z}),\ z \in H^4(\mathbb{HP}^\infty;\mathbb{Z}),\ w \in H^2(\mathbb{RP}^\infty;\mathbb{Z}).$

For $* = 4k+1, \quad 4k+3,$

$$\begin{array}{ccccc}
H^*(Sp^2;\mathbb{Z}) & \xleftarrow{u^*} & H^*(SP^2/N;\mathbb{Z}) & \longleftarrow & H^{*-1}(N;\mathbb{Z}) \\
\downarrow f^* & & r^* \downarrow \cong & & \downarrow \\
H^*(L;\mathbb{Z}) & \longleftarrow & H^*(L/A;\mathbb{Z}) & \xleftarrow{\delta} & H^{*-1}(A;\mathbb{Z})
\end{array}$$

The integral cohomology of $SP^2_n := SP^2(\mathbb{HP}^n)$

Lemma

The homomorphism $\eta^* : H^*(SP^2; \mathbb{Z}) \to H^*(SP^2_n; \mathbb{Z})$ is surjective.

Roughly speaking, $\ker \eta^*$ is explained by the fact

$$z^{n+1} = 0 \quad \text{in} \quad H^*(\mathbb{HP}^n; \mathbb{Z}) \cong \mathbb{Z}[z]/(z^{n+1});$$

for instance,

$$\eta^*(t_{i,j}) = 0, \quad \text{if} \quad i > n.$$

Example: $n = 3$

Example: $H^*(SP^2_3;\mathbb{Z})$

The integral cohomology of SP^2_3 can be expressed as

$$H^*(SP^2_3;\mathbb{Z}) \cong H^*(SP^2;\mathbb{Z})/\mathcal{I}$$

where

$$\mathcal{I} = \left(t_{i,j}, \quad g^4 - 4\cdot\frac{1}{2}g^2h + \frac{1}{2}h^2, \quad \frac{1}{2}g^3h - 3\cdot\frac{1}{4}gh^2, \quad g^7, \quad \frac{1}{8}h^4 \right)$$

for $i > 3$.

Table: n=3

$H^* := H^*(SP^2_3; \mathbb{Z})$ with generators.

$*$	0	4	7	8	11
H^*	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}/2$
	1	g	$t_{1,1}$	g^2, h	$t_{2,1}$

	12	13	15	16	17
	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}/2$
	$g^3, \frac{1}{2}gh$	$t_{2,2}$		$\frac{1}{2}g^2h, \frac{1}{2}h^2$	$t_{3,2}$

	19	20	21	23	24
	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$
	$t_{3,3}$	$\frac{1}{4}gh^2$	$t_{3,4}$	$t_{3,5}$	$\frac{1}{4}h^3$

$H^* = 0$ for other $*$.

Table: $H^*(SP^2_3;\mathbb{Z})$

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	$g^3, \frac{1}{2}gh$	$t_{2,2}$	$t_{2,3}, t_{3,1}$	$\frac{1}{2}g^2h, \frac{1}{2}h^2$	$t_{3,2}$

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$H^* = 0$ for other $*$.

Product structures: Part I

The following product structures on $H^*(SP^2;\mathbb{Z})$ become visible after analysing the LESs.

Some of the product structures

- $t_{i,j}t_{k,l} = 0$,
- $(1/2)^{s-1}h^s \cdot (1/2)^{u-1}h^u = 2 \cdot (1/2)^{s+u-1}h^{s+u}$,
- $(1/2)^m g^\ell h^m \cdot (1/2)^{s-1}h^s = 2 \cdot (1/2)^{m+s}g^\ell h^{m+s}$,
- $(1/2)^m g^\ell h^m \cdot (1/2)^j g^i h^j = (1/2)^{m+j}g^{\ell+i}h^{m+j}$.

However in order to obtain the remaining multiplicative structure some other approach is needed!

Product structures: part II

How to find the products

$$t_{i,j} \cdot (1/2)^m g^\ell h^m = ? \quad \text{and} \quad t_{i,j} \cdot (1/2)^{s-1} h^s = ?$$

Recall $2t_{i,j} = 0$.

Lemma

The mod 2 reduction

$$\rho : H^*(SP^2; \mathbb{Z}) \to H^*(SP^2; \mathbb{Z}/2)$$

is injective in dimensions $* \not\equiv 0 \mod 4$.

Then repeat our method of LESs using additional input.

Cohomology with $\mathbb{Z}/2$ coefficients

And we get...

$$t_{i,j} \cdot (1/2)^{s-1} h^s = 0 \quad \text{and} \quad t_{i,j} \cdot (1/2)^m g^\ell h^m = 0$$

It is generally believed that computation of $\mathbb{Z}/2$-cohomology is relatively easier than integral cohomology (what I was told) !!

I have carried out the calculations, and they agree with Nakaoka's work; they are also consistent with applying the Universal Coefficient Theorem to $H^*(SP^2_n; \mathbb{Z})$.

$\mathbb{Z}/2$-Cohomology

There are several vital requirements for calculating the $\mathbb{Z}/2$- cohomology.

$\mathbb{Z}/2$-cohomology

These include:

- the Thom space SP^2/N
- the Steenrod squares
- $BPin(4)$ and more...

We will talk about these

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- $BPin(4)$ and more...

We will talk about these on 19 August 2014 at ICM in Seoul.

See you in Seoul

Thank you for listening.