Torus fibrations and localization of index

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Purpose

Joint work with Hajime Fujita and Mikio Furuta:

1. H. Fujita, M. Furuta, and T. Y, *Torus fibrations and localization of index I*, J. Math. Sci. Univ. Tokyo 17 (2010), no. 1, 1-26.
2. ______, *Torus fibrations and localization of index II*, Comm. Math. Phys. 326 (2014), no. 3, 585-633.
3. ______, *Torus fibrations and localization of index III*, Comm. Math. Phys. 327 (2014), no. 3, 665-689.

These joint works are concerned with an index theory for a Dirac-type operator on a possibly noncompact Riemannian manifold.

Purpose of this talk

1. *an overview of the index theory for a Dirac-type operator on a possibly noncompact Riemannian manifold*
2. *applications to symplectic geometry*

Motivation

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In the context of the geometric quantization of Lagrangian fibrations and Hamiltonian T-actions, it is often observed

- $RR(M,\omega) =$# *of Bohr-Sommerfeld fibers*
  - $RR(M,\omega)$ $\cdots$ dim *of the quantum Hilbert space of Spin^c quantization*
  - *# of BS* $\cdots$ dim *of the quantum Hilbert space of the geometric quantization using real polarization*
- $RR_T(M,\omega)$ *can be written as the sum of the contributions from the lattice points in the moment map image. (Ex. Danilov's formula concering the equivariant Riemann-Roch index for a projective toric variety)*

One of the motivation is to understand the mechanism underlying these phenomena from the viewpoint of index theory.

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Input data

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$$\overset{\text{def}}{\Leftrightarrow} \begin{cases} W = W^0 \oplus W^1 \\ \exists c\colon TM \to \mathrm{End}_{\mathbb{C}}(W) \text{ such that } c(u)c(v) + c(v)c(u) = -2g(u,v)\,\mathrm{id}_W \end{cases}$$

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- $\pi\colon V \to U$: fiber bundle
  - a fiber is a torus
  - π is Riemannian submersion w.r.t. g and some Riem. metric on U
- $D_{\text{fiber}} \circlearrowright \Gamma(W|_V)$: 1^{st} order formally self-adjoint differential operator of degree-one satisfying
  - D_{fiber} contains only derivatives along fibers of π
  - $\forall b \in U$ $D_{\text{fiber}}|_{\pi^{-1}(b)} \circlearrowright \Gamma\left(W|_{\pi^{-1}(b)}\right)$ is a Dirac-type operator of $\pi^{-1}(b)$
  - $\ker\left(D_{\text{fiber}}|_{\pi^{-1}(b)}\right) = 0$ $\forall b \in U$
  - The Clifford multiplication of TB anti-commutes with D_{fiber}.

Local index theory
Output

Main Theorem (Fujita-Furuta-Y.)

For these input data, $\exists\, \mathrm{ind}(M, V : W) \in \mathbb{Z}$ *satisfying the following properties:*

1. $\mathrm{ind}(M, V : W)$ *is invariant under continuous deformation of the data.*
2. *If M is closed, then* $\mathrm{ind}(M, V : L)$ *is equal to the index of a Dirac-type operator.*
3. *For an open subset $V' \subset V$ with $M \setminus V'$ compact*
$$\mathrm{ind}(M, V : W) = \mathrm{ind}(M, V' : W).$$
4. *Excision property. For an open neighborhood M' of $M \setminus V$*
$$\mathrm{ind}(M, V : W) = \mathrm{ind}(M', V \cap M' : W|_{M'}).$$
5. *Gluing formula*
6. *Product formula*

Outline of Proof

For $t \geq 0$ consider the following perturbation of the Dirac-type operator D

$$D_t := D + t\rho D_{\text{fiber}},$$

where ρ is a cut off function on V. Then one can show

1. If M is compact and $V = M$, then $\ker D_t = 0$ for a sufficiently large t.
2. If $V = N \times (0, \infty)$ and all the data are translation invariant on V. From 1 one can deduce the following
    1. On V $D_t = \alpha(\partial_r + D_{N,t})$. Then, $\ker D_{N,t} = 0$ for a sufficiently large t.
    2. $\dim \ker D_t|_{W^0} \cap L^2 - \dim \ker D_t|_{W^1} \cap L^2$ is independent of sufficiently large t. ($\because$ Atiyah-Patodi-Singer boundary condition)

Definition

$$\operatorname{ind}(M, V : L) := \dim \ker D_t|_{W^0} \cap L^2 - \dim \ker D_t|_{W^1} \cap L^2 \text{ for } \forall t \gg 0.$$

3. For a general case,
    1. Deform V cylindrically so that all the data are translation invariant, and come down to the cylindrical case.
    2. Check the definition is independent of a choice of a cut locus.

Remarks on Main Theorem

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1. *The perturbation used in the proof can be understood as an infinite dimensional analogue of Witten's deformation.*
2. *Theorem can be modified to the following case where*
   - $V = \cup_i V_i$,
   - *each* V_i *is equipped with a torus bundle* $\pi_i \colon V_i \to U_i$,
   - *the ranks of* π_i*'s can vary according to* i,
   - π_i*'s satisfy a certain compatibility condition on* $V_i \cap V_j$.

   It is necessary to formulate a product formula. Such a case will be mentioned later.

Question

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We will explain the following two cases provide such input data.

1. Lagrangian fibrations
2. Hamiltonian S^1-actions

The first Case - Lagrangian fibration

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Definition (Lagrangian fibration)

$f\colon (M,\omega) \to B$ is a Lagrangian fibration $\overset{\text{def}}{\Leftrightarrow} \begin{cases} f\colon \text{fiber bundle} \\ f^{-1}(b)\colon \text{ Lagrangian } \forall\, b \in B \end{cases}$

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Example

$$\begin{array}{cccc} f_0\colon & \left(\mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})^n, \sum_{i=1}^n dx_i \wedge dy_i\right) & \to & \mathbb{R}^n \\ & \Cup & & \Cup \\ & (x_1, \ldots, x_n, y_1, \ldots, y_n) & \mapsto & (x_1, \ldots, x_n) \end{array}$$

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Theorem (Arnold-Liouville)

A Lagrangian fibration $f\colon (M,\omega) \to B$ with closed connected fibers is locally modeled on $f_0\colon \left(\mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})^n, \sum_i dx_i \wedge dy_i\right) \to \mathbb{R}^n$.

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$$\overset{\text{def}}{\Leftrightarrow} \begin{cases} L \to M \ \text{ Hermitian line bundle} \\ \nabla^L \text{ connection of } L \text{ with } \frac{\sqrt{-1}}{2\pi} F_{\nabla^L} = \omega \end{cases}$$

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$f^{-1}(b)$ is *Bohr-Sommerfeld* $\overset{\text{def}}{\Leftrightarrow} H^0\left(f^{-1}(b); (L, \nabla^L)|_{f^{-1}(b)}\right) \neq 0$

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Example

$f_0\colon \left(\mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})^n, \sum_{i=1}^n dx_i \wedge dy_i\right) \to \mathbb{R}^n$

$\left(\mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})^n \times \mathbb{C}, d - 2\pi\sqrt{-1}\sum_{i=1}^n x_i dy_i\right) \to \left(\mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})^n, \sum_{i=1}^n dx_i \wedge dy_i\right)$

Then

$$f_0^{-1}(x) \text{ is BS} \ \Leftrightarrow \ x \in \mathbb{Z}^n.$$

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$$\begin{aligned}\because\quad RR(M,\omega) &= \text{ind}(M, V : W)\ (\because M : \text{closed})\\ &= \text{sum of contributions from BS fibers } (\because \text{excision})\\ &= \#\text{ of BS } (\because \text{computation of 2-dim. cylinder \& product formula})\end{aligned}$$

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Definition (*L*-acyclic orbit)

$$\mathcal{O} : L\text{-acyclic} \overset{\text{def}}{\Leftrightarrow} H^0(\mathcal{O}; (L, \nabla^L)|_{\mathcal{O}}) = 0$$

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Definition (L-acyclic orbit)

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Example

$$(\mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \mathbb{C}, d - 2\pi\sqrt{-1}xdy) \to (\mathbb{R} \times \mathbb{R}/\mathbb{Z}, dx \wedge dy)$$

$$S^1 = \mathbb{R}/\mathbb{Z} \curvearrowright \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \mathbb{C}, \; t(x, y, w) = (x, y + t, e^{2\pi\sqrt{-1}mt} w) \; (m \in \mathfrak{t}^*_{\mathbb{Z}} \cong \mathbb{Z})$$

Then

$$\mathcal{O}_x := \{x\} \times \mathbb{R}/\mathbb{Z} \text{ is } L\text{-acyclic} \Leftrightarrow x \notin \mathbb{Z}.$$

L-acyclic orbit and moment map

For the S^1-action on the prequantum line bundle $(L, \nabla^L) \to (M, \omega)$, the moment map $\mu \colon M \to \mathfrak{t}^*$ ($\mathfrak{t} := \mathrm{Lie}(S^1)$) is defined by the Kostant formula

$$\mathcal{L}_{X_\xi} s = \nabla_{X_\xi} s + 2\pi\sqrt{-1}\,\langle \mu, \xi \rangle\, s$$

for $\forall \xi \in \mathfrak{t}$ and $\forall s \in \Gamma(L)$, where X_ξ is the infinitesimal action of ξ.

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Lemma

Let $\mathcal{O}$ be an orbit. If $\mathcal{O}$ is NON L-acyclic, i.e. $H^0\left(\mathcal{O}; (L, \nabla^L)|_{\mathcal{O}}\right) \neq 0$, then $\mathcal{O} \subset \mu^{-1}(\mathfrak{t}^*_{\mathbb{Z}})$.

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Theorem (Fujita-Furuta-Y)

For above data $\exists\, \mathrm{ind}_{S^1}(M, V : W) \in R(S^1)$ satisfying the properties in Main Theorem. In particular, for each $\gamma \in \mu(M)\cap \mathfrak{t}^*_{\mathbb{Z}}$ let V_γ be a sufficiently small S^1 invariant neighborhood of $\mu^{-1}(\gamma)$ so that $\{V_\gamma\}_{\gamma\in\mu(M)\cap\mathfrak{t}^*_{\mathbb{Z}}}$ are mutually disjoint. Then

$$\mathrm{ind}_{S^1}(M, V : W) = \oplus_{\gamma\in\mu(M)\cap\mathfrak{t}^*_{\mathbb{Z}}} \mathrm{ind}_{S^1}(V_\gamma, V_\gamma \cap V : L|_{V_\gamma}) \in R(S^1).$$

The local index theory for Hamiltonian S^1-actions

Example (cylinder)

$(M,\omega) := ((i-\varepsilon, i+\varepsilon) \times \mathbb{R}/\mathbb{Z}, dx \wedge dy) \quad (0 < \varepsilon < 1, i \in \mathbb{Z})$

$(L, \nabla^L) := \left(M \times \mathbb{C}, d - 2\pi\sqrt{-1}xdy\right) \to (M,\omega)$

$S^1 = \mathbb{R}/\mathbb{Z} \curvearrowright (L, \nabla^L),\ t(x,y,w) = (x, y+t, e^{2\pi\sqrt{-1}mt}w) \quad (m \in \mathfrak{t}^*_{\mathbb{Z}} \cong \mathbb{Z})$

$V := M \setminus \{i\} \times \mathbb{R}/\mathbb{Z}$

Then

$$\mathrm{ind}_{S^1}(M, V : W) = \mathbb{C}_{i-m}$$

Multiplicity

For $\sigma \in \mathfrak{t}_{\mathbb{Z}}^*$ and $U \in R(S^1)$ we denote the multiplicity of the irreducible representation of S^1 with weight σ in U by

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Theorem (Fujita-Furuta-Y)

For each $\gamma \in \mu(M) \cap \mathfrak{t}_{\mathbb{Z}}^*$ and $\sigma \in \mathfrak{t}_{\mathbb{Z}}^*$ with $\gamma \neq \sigma$

$$\mathrm{ind}_{S^1}(V_\gamma, V_\gamma \cap V : L|_{V_\gamma})^\sigma = 0.$$

[Q,R]=0

Suppose $\gamma \in \mu(M) \cap \mathfrak{t}^*_{\mathbb{Z}}$ is a regular value of μ. Then, a new symplectic manifold $(M_\gamma, \omega_\gamma)$ with prequantum line bundle $(L_\gamma, \nabla^{L_\gamma})$ is obtained by

$$(L_\gamma, \nabla^{L_\gamma}) := \left((L, \nabla^L) \otimes \mathbb{C}_\gamma|_{\mu^{-1}(\gamma)}\right) / S^1$$

$$\downarrow$$

$$(M_\gamma, \omega_\gamma) := \left(\mu^{-1}(\gamma), \omega|_{\mu^{-1}(\gamma)}\right) / S^1.$$

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Corollary (Guillemin-Sternberg, Meinrenken, Vergne, Tian-Zhang, . . .)

Let $(L, \nabla^L) \to (M, \omega)$ be as above. Assume M is closed and $\gamma \in \mu(M) \cap \mathfrak{t}^*_{\mathbb{Z}}$ is a regular value of μ. Then

$$RR(M, \omega)^\gamma = RR(M_\gamma, \omega_\gamma).$$

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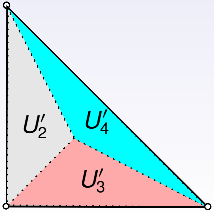

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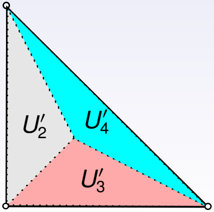

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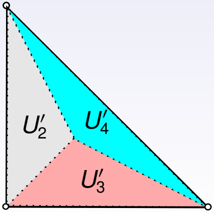

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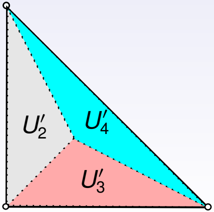

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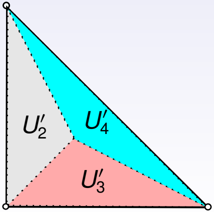

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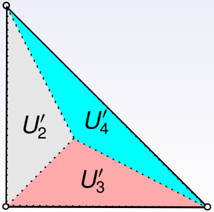

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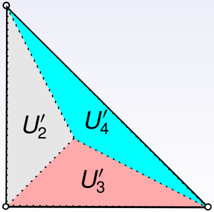

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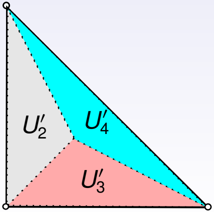
$\overset{\text{induce}}{\Longrightarrow}$ the notion of "compatible fibration and acyclic compatible system" and we can generalize the local index theory for compatible acyclic systems. (These will be explained in the next talk.)



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- By the local index theory we often obtain a localization formula for the index. But, in general, it is difficult to compute the local contributions. The condition “four-dimension" is a technical condition to compute the contribution from singular fibers. The next talk by Fujita may become a breakthrough to get a systematic computation of the contribution from singular fibers.

Thank you for your attention!