The theory of $(2n, k)$-manifolds

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Content of the talk

- Motivating results
- Theory of toric $(2n, k)$ - manifolds
- Complex Grassmann and flag manifolds as $(2n, k)$ - manifolds
- Seminal examples of $(2n, k)$-manifolds

The main ideas of our approach are published in:

Victor M. Buchstaber (joint with Svjetlana Terzić)
$(2n, k)$*-manifolds and applications*, Mathematisches Forschung Institut Oberwolfach, Report No. 27/2014, p. 5–8, DOI: 10.4171/OWR/2014/27

We consider the complex Grassmann manifold $G_{4,2} = U(4)/U(2) \times U(2)$ and the canonical action $T^4 \hookrightarrow G_{4,2}$ which induces effective action of T^3

Theorem

$X = G_{4,2}/T^3$ is homeomorphic to the quotient space

$$(\Delta_{4,2} \times CP^1)/ \approx$$

$$\text{where } (x, c) \approx (y, c^{'}) \Leftrightarrow x = y \in \partial\Delta_{4,2}.$$

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$$\text{where } (x, c) \approx (y, c') \Leftrightarrow x = y \in \partial\Delta_{4,2}.$$

Corollary

$G_{4,2}/T^3$ is homeomorphic to the join $S^2 * S^2$ which is homotopy equivalent to S^5.

Theorem

$G_{4,2}/T^3$ is a topological manifold without boundary, and, thus, $G_{4,2}/T^3$ is homeomorphic to the sphere S^5.

- S^5 has unique differentiable structure, the standard one;
- suggests: no differentiable structure on $X = G_{4,2}/T^3$ such that $\pi : G_{4,2} \to X$ is a smooth map;
  otherwise X would be diffeomoirphic to the standard sphere S^5, $S^1 \hookrightarrow S^5$ smoothly, while it is not clear where such an action on X would come from.

— We prove the quotient structure is not differentiable;

— Describe the corresponding smooth and singular points;

Orbit space $\mathbb{C}P^5/T^3$

Consider representation $T^4 \to T^6$ given by

$$(t_1, t_2, t_3, t_4) \to (t_1 t_2, t_1 t_3, t_1 t_4, t_2 t_3, t_2 t_4, t_3 t_4)$$

and the action $T^4 \hookrightarrow \mathbb{C}P^5$ given as the composition of this representation and the standard action of T^6 on $\mathbb{C}P^5$. We obtain an effective action $T^3 \hookrightarrow \mathbb{C}P^5$

We prove:

Theorem

CP^5/T^3 is homeomorphic to $\partial\Delta_{4,2} * \mathbb{C}P^2 \equiv S^2 * \mathbb{C}P^2$.

The Plücker embedding of $G_{4,2}$ into $\mathbb{C}P^5$ is equivariant under the canonical action of T^4 on $G_{4,2}$ and the action of T^4 on $\mathbb{C}P^5$ given by representation $T^4 \to T^6$ by the second symmetric power. It implies:

$$G_{4,2}/T^3 \subset \mathbb{C}P^5/T^3 : \quad S^2 * \mathbb{C}P^1 \subseteq S^2 * \mathbb{C}P^2, \text{ where } \mathbb{C}P^1 \subset \mathbb{C}P^2$$

$$(c,1) \to (c:1:(1-c)), \quad (1,0) \to (0,0,1).$$

Some properties of the action of T^3 on $G_{4,2}$

- There is the standard moment map $\mu : G_{4,2} \to \mathbb{R}^4$ defined by:

$$\mu(X) = \frac{\sum\limits_J |P^J(X)|^2 \delta_J}{\sum\limits_J |P^J(X)|^2},$$

  where $J \subset \{1,2,3,4\}$, $\|J\| = 2$ and $P^J(X)$ are the Plücker coordinates for $X \in G_{4,2}$ and $\delta_J \in \mathbb{R}^4$ is given by

$$(\delta_J)_i = 1, \quad i \in J, \quad (\delta_J)_i = 0, \quad i \notin J.$$

  This map is invariant for the canonical action of T^4 on $G_{4,2}$ and trivial action on $\mathbb{R}^4$.

- $Im\mu = \Delta_{4,2}$ —octahedron.

- There exists a smooth atlas (M_J, φ_J) for $G_{4,2}$:

$$M_J = \{X \in G_{4,2} \mid P^J(X) \neq 0\}, \quad \varphi_J : M_J \to \mathbb{C}^4.$$

  $X \in M_J \Rightarrow$ it can be represented by matrix A such that $A_J = I_d$ and

$$\varphi_J(X) = (a_{ij}(X)) \in \mathbb{C}^4, \quad i \notin J$$

  Each chart M_J is T^3-invariant, everywhere dense set in $G_{4,2}$ and contains exactly one fixed point which maps to zero by the coordinate map.

- For any chart (M_J, φ_J) it is given the characteristic homomorphism $\alpha_J : T^3 \to T^4$ such that the homeomorphism φ_J is α_J - equivariant:

$$\varphi_J(tm) = \alpha_J(t)\varphi_J(m), \quad t \in T^3, \; m_J \in M_J.$$

- For any characteristic homomorphism $\alpha_J : \mathbb{T}^3 \to \mathbb{T}^4$, the weight vectors are pairwise linearly independent.
- The map μ gives the bijection between the set of fixed points and the set of vertices of the polytope $\Delta_{4,2}$.

$(2n, k)$-manifolds

We assume the following to be given:

- a smooth, closed simply connected manifold M^{2n};
- a smooth, effective action θ of the torus T^k on M^{2n}, where $2 \leq k \leq n$, such that the stabilizer of any point is connected;
- an open θ-equivariant map $\mu : M^{2n} \to \mathbb{R}^k$ whose image is a k-dimensional convex polytope, where $\mathbb{R}^k$ is considered with trivial T^k - action.

- μ - we call an *almost moment map*.
- $Im\mu$ we denote by P^k.
- It is defined the characteristic function

$$\chi : M^{2n} \to S(T^k) \text{ by } \chi(x) = stab(x)$$

- The function χ induces mapping from $M^{2n}/T^k \to S(T^k)$.

Axiom 1:

There is a smooth atlas $\mathfrak{M} = \{(M_i, \varphi_i)\}_{i\in I}$ with the homeomorphisms $\varphi_i : M_i \to \mathbb{R}^{2n} \approx \mathbb{C}^n$ for the fixed identification $\approx$, such that any chart M_i

- is T^k-invariant,
- contains exactly one fixed point x_i with $\varphi_i(x_i) = (0, \ldots, 0)$,
- the closure of M_i is M^{2n}.

Smooth structure

Axiom 1:

There is a smooth atlas $\mathfrak{M} = \{(M_i, \varphi_i)\}_{i\in I}$ with the homeomorphisms $\varphi_i : M_i \to \mathbb{R}^{2n} \approx \mathbb{C}^n$ for the fixed identification $\approx$, such that any chart M_i

- is T^k-invariant,
- contains exactly one fixed point x_i with $\varphi_i(x_i) = (0, \ldots, 0)$,
- the closure of M_i is M^{2n}.

Corollary. The action of T^k on M^{2n} has finitely many isolated fixed points.

Denote by m the number of fixed points for T^k-action on M^{2n}.

The charts given by Axiom 1 we enumerate as $(M_1, \varphi_1), \ldots, (M_m, \varphi_m)$.

The sets $Y_i = \partial M_i = M - M_i$ are closed and T^k-invariant.

Define the sets W_σ, where $\sigma = \{i_1, \ldots, i_l\} \subseteq \{1, \ldots, m\}$ as:

$$W_\sigma = M_{i_1} \cap \cdots M_{i_l} \cap Y_{i_{l+1}} \cap \cdots Y_{i_m},$$

where $\{i_{l+1}, \ldots, i_m\} = \{1, \ldots, m\} - \{i_1, \ldots, i_l\}$.

Definition

The non-empty set W_σ is called admissible and the corresponding set σ is called admissible too.

Lemma

The admissible sets W_σ are T^k-invariant, pairwise disjoint and $M^{2n} = \cup W_\sigma$.

$W_{\{1,\dots,m\}}$ is an admissible set which is everywhere dense in M^{2n}.

$W_{\{i\}}$ is an admissible set for any $1 \leq i \leq m$.

Axiom 2:

The map μ gives the bijection between the set of fixed points and the set of vertices of the polytope P^k.

Admissible sets and polytopes

Axiom 2:

The map μ gives the bijection between the set of fixed points and the set of vertices of the polytope P^k.

Let $S(P^k)$ be the family of convex polytopes which are spanned by the vertices of the polytope P^k and $\{W_\sigma\}$ the family of all admissible sets. Define the map $s : \{W_\sigma\} \to S(P^k)$ by

$$s(W_\sigma) = P_\sigma, \text{ where } \sigma = \{i_1, \ldots, i_l\} \text{ and } P_\sigma = convhull(v_{i_1}, \ldots, v_{i_l}),$$

and $v_{i_1}, \ldots, v_{i_l}$ are the vertices of the polytope P^k determined by

$$v_{i_j} = \mu(x_{i_j}) \text{ for } x_{i_j} \in M_{i_j} - \text{ the fixed point.}$$

Definition

A polytope $P_\sigma \in S(P^k)$ is said to be admissible if it corresponds to an admissible set.

The polytope P^k is an admissible polytope, where $\sigma = \{1, \ldots, m\}$.

For a general $(2n, k)$-manifold two admissible polytopes may intersect.

See for example complex flag manifold F_3 as $(6, 2)$-manifold and complex Grassmann manifold $G_{4,2}$ as $(8, 3)$-manifold.

Definition

A point $p \in P^k$ is said to be singular if $p \in P_{\sigma_1} \cap P_{\sigma_2}$ for some $P_{\sigma_1}, P_{\sigma_2} \in \mathfrak{S}$, thus we obtain the set of singular points.

Denote by $\widehat{\mu} : M^{2n}/T^k \to P^k$ the map induced by the almost moment map μ.

Axiom 3:

The almost moment map μ:

- gives the mapping from W_σ to $\overset{\circ}{P}_\sigma$,
- induces fiber bundle $\widehat{\mu} : W_\sigma/T^k \to \overset{\circ}{P}_\sigma$.

Choose $x_\sigma \in \overset{\circ}{P}_\sigma$ and let $F_\sigma = \widehat{\mu}^{-1}(x_\sigma)$.

Definition

The set F_σ we call the set of parameters of the admissible set W_σ. It is the fiber of the bundle $\widehat{\mu} : W_\sigma / T^k \to \overset{\circ}{P}_\sigma$.

Since $\overset{\circ}{P}_\sigma$ is contractible we obtain:

Corollary. The fiber bundle $\widehat{\mu} : W_\sigma / T^k \to \overset{\circ}{P}_\sigma$ is isomorphic to the trivial bundle. Hence W_σ / T^k is homeomorphic to $\overset{\circ}{P}_\sigma \times F_\sigma$.

The boundary $\partial W_\sigma = \overline{W_\sigma} - W_\sigma$ of an admissible set W_σ is contained in the union of the admissible sets $W_{\tilde{\sigma}}$ for all subsets $\tilde{\sigma} \subset \sigma$.

In the paper of Gel'fand-Serganova (UMN, 1987) it is given the description of the action of T^6 on the Grassmann manifold $G_{7,3}$ from which we deduce the example of our $(2n, k)$-manifold for which

$$\partial W_\sigma \subset \cup_{\tilde{\sigma} \subset \sigma} W_{\tilde{\sigma}}.$$

More precisely, consider in $G_{7,3}$ the point given by a matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & b_1 & c_1 & d_1 \\ 0 & 1 & 0 & a_2 & 0 & c_2 & d_2 \\ 0 & 0 & 1 & a_3 & b_3 & 0 & d_3 \end{pmatrix}$$

Its $(\mathbb{C}^*)^6$-orbit coincides with a admissible set W_σ which contains it. Thus the set F_σ of parameters of W_σ is a point.
The point given by a matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & b_1 & c_1 & 0 \\ 0 & 1 & 0 & a_2 & 0 & c_2 & 0 \\ 0 & 0 & 1 & a_3 & b_3 & 0 & 0 \end{pmatrix}$$

belongs to ∂W_σ. This point belongs to the admissible set $W_{\sigma'}$ and the set of parameters $F_{\sigma'}$ of $W_{\sigma'}$ is two-dimensional.

For a given trivialization $\xi_\sigma : W_\sigma / T^k \to F_\sigma$ and any point $c_\sigma \in F_\sigma$ the leaf $W_\sigma[\xi_\sigma, c_\sigma] \subseteq W_\sigma$ is defined as

$$W_\sigma[\xi_\sigma, c_\sigma] = (\pi^{-1} \circ \xi_\sigma^{-1})(c_\sigma),$$

where $\pi : W_\sigma \to W_\sigma / T^k$ is the projection.

Axiom 4:

For any admissible σ there exists the trivialization $\xi_\sigma : W_\sigma / T^k \to F_\sigma$ such that for any $c_\sigma \in F_\sigma$ the boundary $\partial W_\sigma[\xi_\sigma, c_\sigma]$ of the leaf $W_\sigma[\xi_\sigma, c_\sigma]$ of W_σ is the union of the leafs $W_{\bar{\sigma}}[\xi_{\bar{\sigma}}, c_{\bar{\sigma}}]$ for exactly one $c_{\bar{\sigma}} \in F_{\bar{\sigma}}$, where $P_{\bar{\sigma}}$ runs through the admissible faces for P_σ.

Note. Axiom 4 is motivated by the results of Atiyah, Guillemin-Sternberg and Gel'fand-MacPherson in the case of $(\mathbb{C}^*)^k$-action on M^{2n}.

Let Axiom 4 is satisfied. Since $\mu(\overline{W_\sigma[\xi_\sigma, c_\sigma]}) = P_\sigma$ we obtain:

Lemma

For any $c_\sigma \in F_\sigma$, the boundary $\partial W_\sigma[\xi_\sigma, c_\sigma]$ of the leaf $W_\sigma[\xi_\sigma, c_\sigma]$ of the stratum W_σ is the union of the leafs $W_{\bar\sigma}[\xi_\sigma, c_{\bar\sigma}]$ for exactly one $c_{\bar\sigma} \in F_{\bar\sigma}$, where $P_{\bar\sigma}$ runs through the all faces for P_σ.

Corollary. Any face of an admissible polytope is an admissible polytope.

Corollary. For any pair $P_{\sigma'} \subset P_\sigma$ there exists the map $\xi_{\sigma,\sigma'} : F_\sigma \to F_{\sigma'}$ such that if $P_{\sigma''} \subset P_{\sigma'} \subset P_\sigma$ then $\xi_{\sigma',\sigma''} \circ \xi_{\sigma,\sigma'} = \xi_{\sigma,\sigma''}$.

CW-complex of admissible polytopes

Let $\mathfrak{S}$ denotes the set of admissible polytopes.
Define the operator d on $\mathfrak{S}$ by

$$dP_\sigma \text{ is disjoint union of all proper faces of } P_\sigma.$$

We obtain CW complex $CW(M^{2n}, P^k)$: the vertices of this complex are the vertices of P^k and open cells are $\overset{\circ}{P}_\sigma$ for $P_\sigma \in \mathfrak{S}$. We glue them by induction using the operator d.

There is the canonical map $\widehat{\pi} : CW(M^{2n}, P^k) \to P^k$.

For any $P_\sigma \in \mathfrak{S}$ there is the cell $\overset{\circ}{P'_\sigma}$ in $CW(M^{2n}, P^k)$ such that the map $\widehat{\pi} : \overset{\circ}{P'_\sigma} \to \overset{\circ}{P}_\sigma$ is a homeomorphism.

We obtain the canonical map $g : M^{2n} \to CW(M^{2n}, P^k)$ defined by

$$x \in M^{2n} \Rightarrow \exists ! W_\sigma,\ x \in W_\sigma \Rightarrow \mu(x) \in \overset{\circ}{P}_\sigma,$$

$$\exists ! y \in P_\sigma^{'} \subseteq CW(M^{2n}, P^k),\ \widehat{\pi}(y) = \mu(x).$$

Theorem

The singular points of P^k can be resolved that is almost moment map $\mu : M^{2n} \to P^k$ decomposes as $\mu = \widehat{\pi} \circ g$ for the canonical map $g : M^{2n} \to CW(M^{2n}, P^k)$.

Note. As it is shown we have $\partial W_\sigma \subset \cup_{\tilde{\sigma} \subset \sigma} W_{\tilde{\sigma}}$ for $G_{7,3}$. Consequently in general case the family $\{W_\sigma\}$ does not give CW-complex. Although there are important examples of $(2n, k)$-manifolds for which $CW(M^{2n}, P^k)$ is covered by the CW-complex of admissible sets.

The orbit space M^{2n}/T^k can be described in terms of $CW(M^{2n}, P^k)$, F_σ and $\xi_{\sigma,\sigma'}$:

Theorem

$$M^{2n}/T^k = \cup P_\sigma \times F_\sigma / \approx,$$

where $(x, f_x) \approx (y, f_y)$ *if and only if*

1. $x = y \in P_{\sigma'} \subset P_\sigma$,
2. $f_y = \xi_{\sigma,\sigma'}(f_x)$.

Axiom 5:

- χ is constant on W_σ for any admissible set W_σ,
- If $W_{\sigma'} \subset \overline{W_\sigma}$ then $\chi(W_\sigma) \subset \chi(W_{\sigma'})$.

We call W_σ a stratum if Axiom 5 is satisfied.
Define the function

$$\widehat{\chi} : \mathfrak{S} \to S(T^k) \text{ by } \widehat{\chi}(P_\sigma) = \chi(x),\ x \in W_\sigma.$$

Then $T_\sigma = T^k / \widehat{\chi}(P_\sigma)$ acts freely on W_σ.
By construction we have:

If $P_{\bar{\sigma}}$ is a facet of P_σ then $\widehat{\chi}(P_\sigma) \subseteq \widehat{\chi}(P_{\bar{\sigma}})$.

Moreover it holds:

Proposition

If $P_\sigma \in \mathfrak{S}$ and $P_{\bar{\sigma}}$ is a facet of P_σ then $\widehat{\chi}(P_\sigma) \subset \widehat{\chi}(P_{\bar{\sigma}})$.

Theorem

Codim $\chi(W_\sigma) = \dim T_\sigma = \dim P_\sigma$ for any stratum W_σ.

Corollary. $\dim W_\sigma[\xi_\sigma, c_\sigma] = 2\dim P_\sigma$ for any admissible polytope P_σ and any $c_\sigma \in F_\sigma$

Corollary. $\dim F_\sigma$ is even.

Almost standard action

Consider an action θ of the torus $\mathbb{T}^k$ on $\mathbb{C}^n$ given by a representation $\rho : \mathbb{T}^k \to \mathbb{T}^n$ and the standard action of $\mathbb{T}^n$ on $\mathbb{C}^n$.

Definition

An action θ is called almost standard if:

1. it is effective,
2. the origin is the only fixed point,
3. the stabilizer of any point $x \in \mathbb{C}^n$ is connected.
4. its weight vectors are pairwise linearly independent.

Remark. For $k = n$ an almost standard action is isomorphic to the standard one.

- The representation ρ can be written as $\rho = (\rho_1, \ldots, \rho_n)$, where $\rho_i : \mathbb{T}^k \to S^1$, $1 \leq i \leq n$.
- The characters ρ_i can be represented as $\rho_i(t) = e^{2\pi\sqrt{-1}\langle \Lambda_i,\, t\rangle}$, where $\Lambda_i \in \mathbb{Z}^k$ are the weight vectors for the representation ρ.

- We obtain the matrix V be a $(k \times n)$-matrix whose rows are given by the weight vectors Λ_i.
- Denote by $P^J(V)$ the Plücker coordinates of the matrix V, where $J \subseteq \{1, \dots, n\}$ and $|J| = k$.
- The matrix V gives the linear map $\mathbb{R}^k \to \mathbb{R}^n$.
- Furthermore, for any subset $J \subseteq \{1, \dots, n\}$, the matrix V^J defined by the vectors $\Lambda_j, j \in J$ gives the linear map $f_J : \mathbb{R}^k \to \mathbb{R}^J$.

Proposition

If the map $f_J : \mathbb{R}^k \to \mathbb{R}^J$ is induced by an almost standard action of $\mathbb{T}^k$ on $\mathbb{C}^n$ then the image $f_J(\mathbb{Z}^k)$ is a direct summand in $\mathbb{Z}^J$ for any $J \subseteq \{1, \dots, n\}$.

Corollary. The Plücker coordinates $P^J(V) \in \{-1, 0, 1\}$.

Corollary. The weight vectors Λ_i, $1 \leq i \leq n$ are primitive.

The action - locally almost standard

Axiom 6:

For any chart (M_i, φ_i) it is given the characteristic homomorphism $\alpha_i : \mathbb{T}^k \to \mathbb{T}^n$ such that:

1. its weight vectors are pairwise linearly independent.
2. the homeomorphism φ_i is α_i - equivariant:

$$\varphi_i(tx_i) = \alpha_i(t)\varphi_i(x_i), t \in \mathbb{T}^k, \; x_i \in M_i.$$

Lemma.

Any characteristic homomorphism $\alpha_i : \mathbb{T}^k \to \mathbb{T}^n$ gives an almost standard action of $\mathbb{T}^k$ on $\mathbb{C}^n$.

Corollary.

The number of fixed points $m \geq k + 1$.

Almost standard action of $(\mathbb{C}^*)^k$ on $\mathbb{C}^n$

Consider an action of the algebraic torus $(\mathbb{C}^*)^k$ on $\mathbb{C}^n$. It induces the action of the compact torus $\mathbb{T}^k$ on $\mathbb{C}^n$ given by a representation $\rho : \mathbb{T}^k \to \mathbb{T}^n$ and the standard action of $\mathbb{T}^n$ on $\mathbb{C}^n$.

Definition

An action of $(\mathbb{C}^*)^k$ on $\mathbb{C}^n$ we call almost standard action if the induced action of $\mathbb{T}^k$ on $\mathbb{C}^n$ is almost standard.

Lemma

For an almost standard action of $(\mathbb{C}^*)^k$ on $\mathbb{C}^n$ it holds

- any one-dimensional $(\mathbb{C}^*)^k$-orbit is one of the coordinate axis,
- for any codimension one subgroup $H < \mathbb{T}^k$ the fixed point set $(\mathbb{C}^n)^H$ is either the origin either it is one of the coordinate axis.

For an almost standard action of $(\mathbb{C}^*)^k$ on $\mathbb{C}^n$ we have the map from the set of coordinate axis to the set of codimension one subgroups of T^k.

The set of one-dimensional orbits

Let H be a codimension one subgroup of T^k. Then

$$(M^{2n})^H = \cup_{1\leq i\leq m} M_i^H, \quad S^1 = T^k/H \text{ acts smoothly on } (M^{2n})^H.$$

Denote by X^H a connected component of $(M^{2n})^H$.
Then X^H is a closed submanifold in M^{2n} and S^1 acts smoothly on X^H.

Proposition

- *X^H is either a fixed point or it is homeomorphic to the sphere S^2 equipped with S^1-action with the fixed point $\{x_i, x_j\}$.*
- *For $X^H \cong S^2$ it holds $X^H - \{x_i, x_j\} \subseteq W_{\{i,j\}}$ and X^H is given by the closure of the preimage of the coordinate axis in the chart M_i as well as the corresponding axis in the chart M_j.*

Corollary

The closure of the set of points in M^{2n} which have one-dimensional orbits is given by the union of $\frac{n \cdot m}{2}$ spheres S^2.

Corollary

$\mu(X^H) = [v_i, v_j]$

Note: If n is odd then the number of fixed points m must be even.

Height function of a $(2n, k)$-manifold

Definition

A linear map $h : \mathbb{R}^k \to \mathbb{R}$, $h(x) =< x, \nu >$ is said to be the height function for T^k manifold M^{2n} if:

1. $h(v_i) \neq h(v_j)$ for any two vertices v_i and v_j of P^k,
2. the composition $h \circ \mu : M^{2n} \to \mathbb{R}$ is a Morse function whose critical points coincides with the fixed points for T^k-action on M^{2n}.

Remark The condition for $h \circ \mu$ to be a Morse function does not depend on the vector ν in general position.

Axiom 7

For a $(2n, k)$-manifold there is a height function $h : \mathbb{R}^k \to \mathbb{R}$.

Graph of a $(2n, k)$-manifold

Definition

Graph $\Gamma(M^{2n}, P^k)$ of $(2n, k)$-manifold M^{2n} is a graph given by the vertices and 1-dimensional admissible polytopes of P^k.

It is 1-skeleton of the complex $CW(M^{2n}, P^k)$.
At any vertex of the graph $\Gamma(M^{2n}, P^k)$ there are exactly n edges.

The height function produces the orientation of the graph $\Gamma(M^{2n}, P^k)$.

Betti numbers of a $(2n, k)$-manifold

- The index $ind(v)$ of the vertex v of the graph $\Gamma(M^{2n}, P^k)$ is the number of edges of $\Gamma(M^{2n}, P^k)$ incoming into vertex v.
- We denote by h_q the number of vertices of $\Gamma(M^{2n}, P^k)$ having index q.

Theorem

The number h_q is equal to $2q$-th Betti number for M^{2n}:

$$h_q = b_{2q}(M^{2n}), \quad q = 0, \ldots, n.$$

The classical Poincare duality theorem gives:

Corollary

$h_q = h_{n-q}, \quad q = 0, \ldots, n.$

Labeled graphs

Let us consider a graph Γ with the set of vertices V and the set of edges E.

Definition

A graph Γ is called $\mathbb{Z}^k$-labeled if it is fixed a mapping $l : E \to \mathbb{Z}^k$.

Definition

A mapping $s : V \to \mathbb{Z}[t]$, where $t = (t_1, \ldots, t_k)$ is said to be suitable if $s(v_1) - s(v_2)$ is divisible by the linear form $\langle l(r), t\rangle$ for any edge r which connects the vertices v_1 and v_2.

Definition

GKM(Γ, l) -ring of the labeled graph (Γ, l) is the ring of all suitable maps $s : V \to \mathbb{Z}[t]$, with the pointwise multiplication.

Labeled graph of a $(2n, k)$-manifold

Let E - denotes the set of edges of the graph $\Gamma(M^{2n}, P^k)$. We have the function $\widehat{\chi} : E \to S(T^k)$ and

$$\dim \widehat{\chi}(e) = k - 1, \quad e \in E.$$

Let $S^1_e = T^k / \widehat{\chi}(e)$, $e \in E$.
The projection $\rho_e : T^k \to S^1_e$ is given by the vector $l_e \in \mathbb{Z}^k$.

The labeled graph of T^k-manifold M^{2n} is the graph $\Gamma(M^{2n}, P^k)$ together with the labeling of its edges given by $e \to l_e$.

This construction is motivated by the construction of GKM graph.

Equivariant cohomology ring of $(2n, k)$-manifolds

Let $B(2n, k) = BT^k \times_{T^k} M^{2n}$ be the Borel construction of a $(2n, k)$-manifold.
The equivariant cohomology for M^{2n} are defined by

$$H^*_{T^k}(M^{2n}, \mathbb{Z}) = H^*(B(2n, k), \mathbb{Z}).$$

Theorem

The GKM -ring of the oriented labeled graph $\Gamma(M^{2n}, P^k)$ of a $(2n, k)$-manifold M^{2n} is isomorphic to $H^*_{T^k}(M^{2n}, \mathbb{Z})$.

Complex Grassmann and flag manifolds

Let $G_{p,q}$ be a complex Grassmann manifold of q-dimensional subspaces in $\mathbb{C}^p$ with canonical action of the compact torus T^p. There is the almost moment map $\mu : G_{p,q} \to \Delta_{p,q}$, where $\Delta_{p,q}$ is the hypersimplex of the dimension $p-1$.
Let F_p be a complex flag manifold of complete flags in $\mathbb{C}^p$ with canonical action of T^p. The manifold F_p has the almost moment map $\mu : F_p \to P_e^{p-1}$, where P_e^{p-1} is the permutahedron.

Theorem

Any $G_{p,q}$ and F_p are $(2n,k)$-manifolds, where $k = p-1$.

Manifold $G_{4,2}$

$$\mu(G_{4,2}) = \Delta_{4,2} \quad - \quad \text{octahedron}$$

$\Delta_{4,2} \subset \mathbb{R}^4$ has 6 vertices and they have two coordinates equal to 1 and two equal to 0.
We list the vertices by the indices of the coordinates equal to 1, that is $12, 13, 14, 23, 24, 34$.

The admissible polytopes are:

1. $\Delta_{4,2}$;
2. 6 four-sided pyramid;
3. 3 diagonal squares;
4. any face on the boundary for $\Delta_{4,2}$.

$W_{\Delta_{4,2}}$ is eight-dimensional stratum with the free action of T^3.

$$W_{\Delta_{4,2}}/T^3 \cong \overset{\circ}{\Delta}_{4,2} \times F_{\Delta_{4,2}}$$

where $F_{\Delta_{4,2}} = (\mathbb{C} - \{0, 1\})$.

For the 6-dimensional strata with admissible polytope P_σ all F_σ are the points. We describe F_σ in terms of $\overline{F_{\Delta_{4,2}}}$.

- $F_\sigma = \{0\}$ for $\sigma = \{12, 13, 23, 24, 34\}$ or $\sigma = \{12, 13, 14, 24, 34\}$;
- $F_\sigma = \{\infty\}$ for $\sigma = \{12, 14, 23, 24, 34\}$ or $\sigma = \{12, 13, 14, 23, 34\}$;
- $F_\sigma = \{1\}$ for $\sigma = \{13, 14, 23, 24, 34\}$ or $\sigma = \{12, 13, 14, 23, 24\}$.

The height function $h : \mathbb{R}^4 \to \mathbb{R}$ is given by $h(x_1, x_2, x_3, x_4) = x_1 + 2x_2 + 4x_3 + 8x_4$.

Complex projective space $\mathbb{C}P^5$ as $(10, 3)$-manifold

Embedding of $G_{4,2}$ into $\mathbb{C}P^5$ given by the Plucker coordinates is T^4-invariant and it is embedding of $(8, 3)$-manifold into $(10, 3)$-manifold. Almost moment map $G_{4,2} \to \Delta_{4,2}$ decomposes into the composition of this embedding and the almost moment map $\mathbb{C}P^5 \to \Delta_{4,2}$.

Any polytope spanned by some subset of vertices for $\Delta_{4,2}$ is admissible polytope for $(10, 3)$-manifold $\mathbb{C}P^5$.

Manifold F_3

F_3 - flag manifold in three-dimensional complex space
It is 6-dimensional manifold with effective action of the compact torus T^2.
There is the almost moment map $\mu : F_3 \to \mathbb{R}^2$.
$\mathrm{Im}\mu = P^2$ is a 6-gon - let us enumerate its vertices as $1, \dots, 6$ anticlockwise.
We obtain $(6, 2)$-manifold and $CW(F_3, P^2)$ has

- 6 vertices: $\{1, \dots, 6\}$
- 9 edges: $[1, 2], [2, 3], \dots, [5, 6], [1, 4], [2, 5], [3, 6]$
- 4 two dimensional cells: $[1, \dots, 6]$, $[1, 2, 3, 4]$, $[3, 4, 5, 6]$ and $[1, 2, 5, 6]$.

The vertices of the corresponding graph have the indices: $0, 1, 1, 2, 2, 3$.
It follows that the h-numbers are: $1, 2, 2, 1$, which are the Betti numbers of F_3.