On the classification of toric varieties

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Toric variety

- toric variety : a normal complex algebraic variety with algebraic $(\mathbb{C}^*)^n$-action having a dense orbit
- toric manifold : a compact non-singular toric variety

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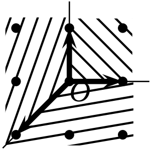
Theorem (Fundamental theorem for toric varieties)

The category of toric varieties is equivalent to the category of fans.

$$X_\Delta \longleftrightarrow \Delta = (K_\Delta, \lambda_\Delta)$$

K_Δ *: simplicial complex,* $\lambda_\Delta : V(K_\Delta) \to \mathbb{Z}^n$

$\Delta =$ 

$$K_\Delta = \{\emptyset, 1, 2, 3, 12, 23, 31\}$$

$$\lambda_\Delta : 1, 2, 3 \mapsto (1,0), (0,1), (-1,-1)$$

Definition

A (non-singular) characteristic map λ on K is a map

$$\lambda\colon V(K) = [m] \to \mathbb{Z}^n$$

s.t. $\{\lambda(i) \mid i \in \sigma\}$ spans unimodular submodule of $\mathbb{Z}^n$ of rank $|\sigma|$ $\forall \sigma \in K$.

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Moreover, λ is fan-giving if $\exists \Delta$ s.t. $(K, \lambda) = (K_\Delta, \lambda_\Delta)$.

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Note : toric manifolds correspond to complete non-singular fans.

Completeness $\Rightarrow K$: star-shapred (simplicial sphere)

Topological analogues of toric manifolds

- Toric manifold $\leftrightarrows (K, \lambda)$ s.t. λ is fan-giving (K is star-shaped)

[1] M. W. Davis and T. Januszkiewicz, Convex polytopes, Coxeter orbifolds and torus actions, Duke Math. J. 62 (1991)

[2] H. Ishida, Y. Fukukawa, M. Masuda, Topological toric manifolds, Moscow Math. J. 13 (2013)

Topological analogues of toric manifolds

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$T^n := (S^1)^n \subset (\mathbb{C}^*)^n$: (compact real) torus of dimension n

Recall that $(\mathbb{C}^*)^n$ acts on X_Δ, T^n also acts on X_Δ.

- A quasitoric manifold M of $\dim 2n$ if $M \circlearrowleft T^n$ locally standard, and $M/T^n \cong P^n$ simple polytope.
  $\leftrightarrows (K, \lambda)$ s.t. K is polytopal. [1]

Recall that a toric manifold admits an algebraic $(\mathbb{C}^*)^n$-action.

- A topological toric manifold is a closed smooth $2n$-manifold M with an effective smooth $(\mathbb{C}^*)^n$-action with some condition.
  $\leftrightarrows (K, \lambda)$ s.t. K is star-shaped (as T^n-manifold). [2]

[1] M. W. Davis and T. Januszkiewicz, Convex polytopes, Coxeter orbifolds and torus actions, Duke Math. J. 62 (1991)

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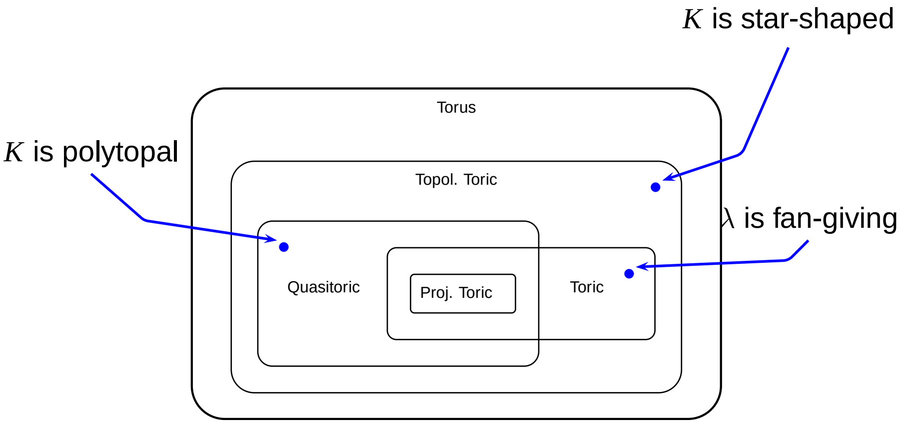
Torus symmetries



Real analogues

- M : a toric variety of complex dimension n.
  Then, $\exists$ a canonical involution ι on M, and M^ι form a real subvariety of real dimension n, called a real toric variety
- Similarly, "real" versions of topol. toric and quasitoric manifolds are real topological toric manifolds and small covers, resp.

Such real analogues of toric objects can be described as a $\mathbb{Z}_2$-version of (K, λ), that is, $\lambda\colon V(K) \to \mathbb{Z}_2^n$.

D-J Equivalence

K : star-shaped with $V(K) = [m]$ and $\lambda\colon V(K) \to \mathbb{Z}^n$
Denote λ by an $n \times m$ matrix

$$\lambda = \left(\begin{array}{cccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \cdots & \color{red}{m} \\ \hline \lambda(1) & \lambda(2) & \lambda(3) & \lambda(4) & \cdots & \lambda(m) \end{array} \right)$$

<u>Note</u> : TFAE

- $M(K,\lambda)$ is Davis-Januszkiewicz equivalent to $M(K,\lambda')$
- $\exists$ weakly equivariant $f\colon M(K,\lambda) \to M(K,\lambda')$ s.t.

$$\begin{array}{ccc} M(K,\lambda) & \xrightarrow{\quad f \quad} & M(K,\lambda') \\ & \searrow \quad \swarrow & \\ & M(K,\lambda)/T^n & \end{array}$$

- λ' can be obtained from λ by elementary row operations.

Classification

To classify toric objects, it seems natural to classify K supporting a toric object first, and find all available λ on fixed K (up to D-J equivalence) in each category.

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1. **Find all K which support toric objects.**
2. **Find all available λ on K when K supports a toric object.**

1. Torus Symmetries
   - Toric objects

2. **Which complexes support toric objects?**
   - Wedge operation
   - Seed

3. Find all available chr. ftn on $K(J)$
   - Projected characteristic map
   - Puzzle

4. Applications
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Wedge operation

K : simplicial complex on $V = [m]$ and $J = (j_1, \ldots, j_m) \in \mathbb{N}^m$.

Wedge operation

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Note : K is determined by its minimal non-faces.

Denote by $K(J)$ the simplicial complex on $j_1 + \cdots + j_m$ vertices

$$\{\underbrace{1_1, 1_2, \ldots, 1_{j_1}}, \underbrace{2_1, 2_2, \ldots, 2_{j_2}}, \ldots, \underbrace{m_1, \ldots, m_{j_m}}\}$$

with minimal non-faces

$$\{\underbrace{(i_1)_1, \ldots, (i_1)_{j_{i_1}}}, \underbrace{(i_2)_1, \ldots, (i_2)_{j_{i_2}}}, \ldots, \underbrace{(i_k)_1, \ldots, (i_k)_{j_{i_k}}}\}$$

for each minimal non-face $\{i_1, \ldots, i_k\}$ of K.

The simplicial wedge operation or (simplicial) wedging of K at i is

$$\mathrm{wedge}_i(K) = K(1, \ldots, 1, 2, 1, \ldots, 1).$$

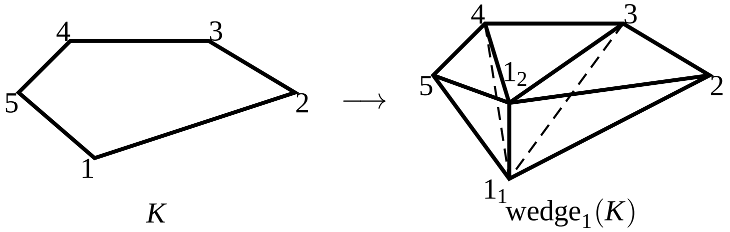
Example

K : the boundary complex of a pentagon.
Then the minimal non-faces of K are

$$\{1,3\}, \{1,4\}, \{2,4\}, \{2,5\}, \text{ and } \{3,5\}.$$

Hence, the minimal non-faces of $\mathrm{wedge}_1(K) = K(2,1,1,1,1)$ are

$$\{1_1, 1_2, 3\}, \{1_1, 1_2, 4\}, \{2,4\}, \{2,5\}, \text{ and } \{3,5\}.$$



Seed

Observe :
Let K be a star-shaped simplicial sphere of dim $n-1$ with m vertices.

$m-n \leq 3$: K supports toric objects $\Leftrightarrow K = K'(J)$, where K' is cross-polytopes, a pentagon, a cyclic polytope $C^4(7)$. [3] [4]

[3] J. Gretenkort, P. Kleinschmidt, B. Sturmfels, *On the existence of certain smooth toric varieties*, Biscrete Comput. 5 (1990)

[4] N. Yu. Erokhovets, *The Buchshtaber invariant of simple polytopes*, Russian Math. Surveys 63 (2008)

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Definition

A star-shaped simplicial complex is called a seed if it cannot be written as a simplicial wedge.

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Proposition

K supports a toric object in each category if and only if so does $K(J)$.

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What we need is to find all "seed" which supports toric objects in each category! However, there are still many seeds.

[3] J. Gretenkort, P. Kleinschmidt, B. Sturmfels, *On the existence of certain smooth toric varieties*, Biscrete Comput. 5 (1990)

[4] N. Yu. Erokhovets, *The Buchshtaber invariant of simple polytopes*, Russian Math. Surveys 63 (2008)

Finiteness of seeds

Let K be a star-shaped simplicial sphere of dim $n-1$ on $[m]$

Proposition

K admits a $\mathbb{Z}_2$-characteristic map $\Leftrightarrow \exists\, \phi\colon [m] \to \mathbb{Z}_2^{m-n}$ s.t. $\{i_1, \dots, i_n\} \in K$ iff $\{\phi(i) \mid 1 \le i \le m,\ i \ne i_k$ for $1 \le k \le n\}$ is a $\mathbb{Z}_2$-basis.

Suppose K supports a $\mathbb{Z}_2$-chr map. If $m \geq 2^{m-n}$, the map ϕ cannot be one-to-one. Suppose that $v, w \in [m]$ and $\phi(v) = \phi(w)$. Then every facet of K should contain either v or w.

Lemma

For $v, w \in [m]$, if every facet of K contains either v or w, then K is equivalent to $L * \partial I$ or $\mathrm{wedge}_v L$ for some simplicial complex L.

Finiteness of seeds

Theorem

K supports a toric object and is a seed $\quad\Longrightarrow\quad m \leq 2^{m-n}$.

<u>**Note**</u> : The "equality" holds only when $m - n \leq 2$.

Corollary

For fixed $m - n$, there are finitely many seeds.

Finiteness of seeds

Theorem

K supports a toric object and is a seed $\implies$ $m \leq 2^{m-n}$.

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Corollary

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Example

$m - n = 3 \quad \Rightarrow \quad m \leq 2^{m-n} - 1 = 7$.

- $n = 2$: K is a pentagon P_5.
- $n = 3$: There are 2 simplicial spheres. Only one $\partial(I^1)^3$ is a seed.
- $n = 4$: There are 5 simplicial spheres. Only one $C^4(7)$ is a seed.

Finiteness of seeds

Example

$m - n = 4 \quad \Rightarrow \quad m \le 2^{m-n} - 1 = 15.$

- $n = 2$: K is a hexagon P_6.
- $n = 3$: There are 5 simplicial spheres. Two of them are seeds.
- $n = 4$: There are 39 simplicial spheres. [a] 24 of them are seeds. 22 of them support toric objects.
- $n = 5, \ldots, 15$: ?.

[a] B. Grunbaum, V.Sreedharan, An enumeration of simplicial 4-polytopes with 8 vertices, J. Combi. Theory (1967)

1. Torus Symmetries
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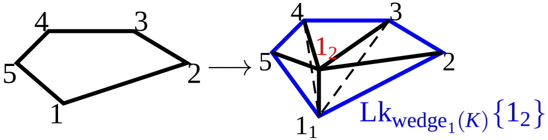
Projected characteristic map

Let K be a simplicial complex and $\sigma \in K$.
The link of σ is

$$\mathrm{Lk}_K\,\sigma := \{\tau \in K \mid \sigma \cup \tau \in K,\ \sigma \cap \tau = \varnothing\}.$$

Note : $\mathrm{Lk}_{\mathrm{wedge}_v K}\{v_1\} \cong K$



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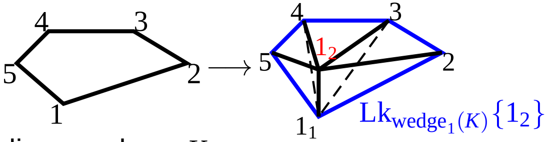

Let (K, λ) be a char. map of dim n and $\sigma \in K$.

Definition

A characteristic map $(\mathrm{Lk}_K\,\sigma, \mathrm{Proj}_\sigma\,\lambda)$, called the projected characteristic map, is defined by the map

$$(\mathrm{Proj}_\sigma\,\lambda)(v) = [\lambda(v)] \in \mathbb{Z}^n / \langle \lambda(w) \mid w \in \sigma \rangle \cong \mathbb{Z}^{n-|\sigma|}.$$

Note : $(\mathrm{Lk}_K\,\sigma, \mathrm{Proj}_\sigma\,\lambda)$ is a char. submanifold of (K, λ).

Classification

K : star-shaped simplicial sphere

Theorem

Assume that λ_1 and λ_2 are char. maps over $\mathrm{wedge}_v(K)$ s.t. $\mathrm{Proj}_{v_i} \lambda_1 = \mathrm{Proj}_{v_i} \lambda_2$ for $i = 1, 2$. Then λ_1 is D-J equivalent to λ_2.

Theorem

- *$\mathrm{wedge}_v(K)$ is polytopal $\Longleftrightarrow$ K is polytopal.*
- *λ is fan-giving $\Longleftrightarrow$ both $\mathrm{Proj}_{v_1} \lambda$ and $\mathrm{Proj}_{v_2} \lambda$ are fan-giving.*

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Theorem

- $\text{wedge}_v(K)$ *is polytopal* $\iff$ K *is polytopal.*
- λ *is fan-giving* $\iff$ *both* $\text{Proj}_{v_1} \lambda$ *and* $\text{Proj}_{v_2} \lambda$ *are fan-giving.*

Roughly speaking.. if one knows every (real) toric object over K, then we know every (real) toric object over $\text{wedge}_v(K)$.

Local information

Definition

The pre-diagram $D'(K)$ of K is an edge-colored multi-graph:

1. the vertex set V whose elements are the D-J classes of K,
2. the edge set E whose elements are $\{\lambda_1, \lambda_2\}$ with color $v \in V(K)$, where $\exists$ a chr map λ over $\mathrm{wedge}_v K$ s.t. $\mathrm{Proj}_{v_i} \lambda = \lambda_i$ for $i = 1, 2$.

$$
\begin{array}{ccc}
\lambda_{11} & \overset{v}{\text{———}} & \lambda_{12} \\
\Big| \, w & & \Big| \, w \\
\lambda_{21} & \overset{v}{\text{———}} & \lambda_{22}
\end{array}
$$

Definition

A square in the figure is realizable if $\exists$ a chr map λ over $\mathrm{wedge}_w(\mathrm{wedge}_v K)$ s.t. $\mathrm{Proj}_{w_i}(\mathrm{Proj}_{v_j} \lambda) = \lambda_{ij}$ for $i, j = 1, 2$.

Puzzle

Let K be a seed with m vertices $[m]$, and $J = (j_1, \dots, j_m) \in \mathbb{N}^m$. Consider the 1-skeleton G of $\Delta^{j_1-1} \times \cdots \times \Delta^{j_m-1}$. Then, each edge e is of the form

$$e = v_1 \times \cdots \times v_{i-1} \times e_i \times v_{i+1} \times \cdots \times v_m,$$

where v_k is a vertex of Δ^{j_k-1}, and e_i is an edge of Δ^{j_i-1}. Then we may color this edge by i.

Theorem

There is 1-1 correspondence between the set of D-J classes over $K(J)$ and the set of graph maps φ from G to $D'(K)$ satisfying

1. *φ preserves the color of edge.*
2. *The image of every square of G is realizable.*

Classification

Definition

A *diagram* $D(K)$ for a seed K is $D'(K)$ equipped with the information of realizable squares.

Corollary

The D-J classes over $K(J)$ is completely determined by $D(K)$.

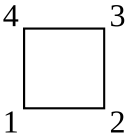
Two edges e and f of G are parallel if

$$e = v_1 \times \cdots \times v_{i-1} \times e_i \times v_{i+1} \times \cdots \times v_m, \quad \text{and}$$
$$f = v'_1 \times \cdots \times v'_{i-1} \times e_i \times v'_{i+1} \times \cdots \times v'_m$$

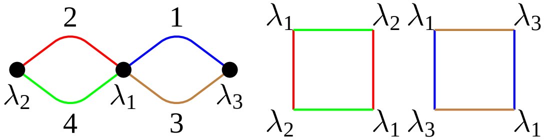
Proposition

If $\varphi(e)$ collapses, then the image of all edges parallel to e collapse.

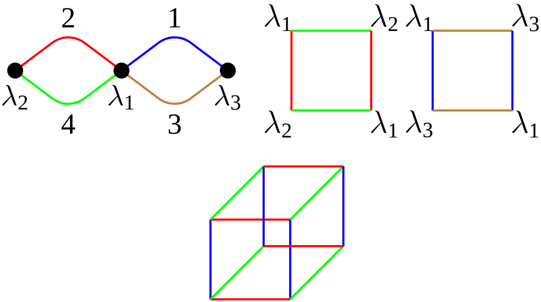
K : 4-gon, $\lambda\colon [4] \to \mathbb{Z}_2^2$ chr map



$$\lambda_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$



Over $K(2,2,1,2)$,



- all edges collapse : 3 small covers
- two sets of parallel edges collapse : 3×2 small covers
- one set of parallel edges collapse : 2 small covers
- no edges collapse : 0 small covers

More generally, over $K(a+1, b+1, c+1, d+1)$,

there are $3 + (2^{a+c} - 2) + (2^{b+d} - 2)$ small covers.

Cyclic polytope $C^4(7)$

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$K = C^4(7)$ is a seed.

$$\lambda_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \text{ and } \lambda_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

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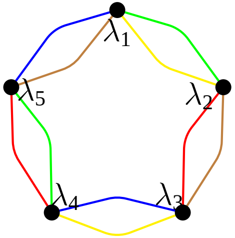
Therefore, $\#(K(J)) = 2$ for any J.

Pentagon

$K = P_5$ is a seed.

$$\lambda_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \quad \text{and} \quad \lambda_5 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$



$$\#DJ(P_5(a,b,c,d,e)) = 2^{a+c-1} + 2^{b+d-1} + 2^{c+e-1} + 2^{d+a-1} + 2^{e+b-1} - 5.$$

Toric manifolds with small Picard number

By applying above way to the category of toric manifolds, we obtains the following corollaries.

Corollary

We classify all toric manifolds of Picard number 3.

Note : It reproves the result of Batyrev[5] without using the fact that such objects are projective.[6]

Corollary

All toric manifolds of $\mathrm{Pic}\,3$ *are projective.*

Note : The above result improves the proof by Kleinschmidt-Stumfels.

[5] V. V. Batyrev, On the classification of smooth projective toric varieties, Tohoku Math. J. (2) 43 (1991)

[6] P. Kleinschmidt, B. Sturmfels, Smooth toric varieties with small picard number are projective, Topology 30 (1991)

Conclusion

In order to classify toric objects (in any category) with $m - n = k$,

1. Find all seeds K. (There are finitely many seeds.)
2. Find all available λ over K and compute $D(K)$. (Local information)
3. Using the puzzle, we construct all available λ over $K(J)$.