Explicit triangulation of complex projective spaces

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Notations

n-simplex:

$$\triangle^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n \leq 1 \text{ and } x_i \geq 0\}$$

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$$\mathcal{F}(\triangle^n) = \{F_i \in \mathcal{L}(\triangle^n) : V(F_i) = \{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}\}$$

Triangulation of manifolds

Definition

A geometrical cell complex K is a finite collection of cells in some $\mathbb{R}^n$ satisfying,

$$(i) \text{ if } \sigma \text{ is a face of } \tau \text{ and } \tau \in K \text{ then } \sigma \in K,$$

$$(ii) \text{ if } \sigma, \tau \in K \text{ then } \sigma \cap \tau \text{ is a face of both } \sigma \text{ and } \tau.$$

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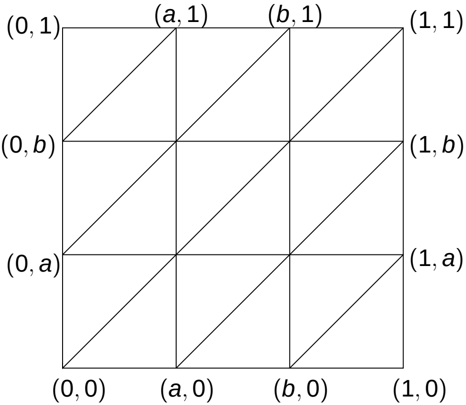
A cell complex K is simplicial if each $\sigma \in K$ is a simplex.

A simplicial complex with 16-vertices



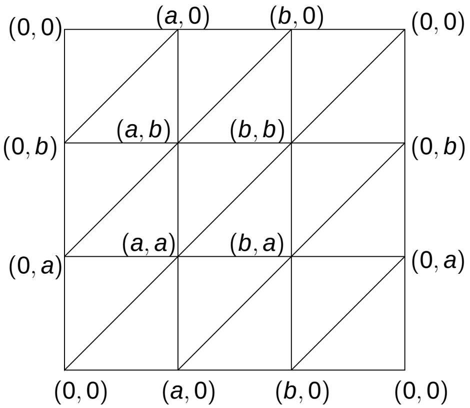
Definition

A simplicial complex K is called a triangulation of a manifold M if

$$|K| = \bigcup_{\sigma \in K} \sigma$$

is homeomorphic to M.

A triangulation of T^2 with 9-vertices



Known triangulation of $\mathbb{C}P^n$

T. F. Banchoff and W. Kuhnel (1983):

A vertex minimum triangulation of $\mathbb{C}P^2$ contains 9 vertices.

Known results on triangulation of $\mathbb{C}P^n$

P. Arnoux and A. Marin (1991):

Lower bound on number of vertices of a triangulation of $\mathbb{C}P^n$ is $1 + \frac{(n+1)^2}{2}$ if $n \geq 3$.

Known triangulation of $\mathbb{C}P^n$

B. Bagchi and B. Datta (2012):

$\mathbb{C}P^3$ was first triangulated with 30 and then 18 vertices.

Theorem (2014)

There is a triangulation of $\mathbb{C}P^n$ *with* $\frac{4^{n+1}-1}{3}$ *vertices for* $n \geq 1$.

Standard basis of $\mathbb{Z}^n$: $\{e_1, \ldots, e_n\}$
$e_0 = e_1 + \cdots + e_n$.

Davis and Januszkiewicz (1991) construction of $\mathbb{C}P^n$

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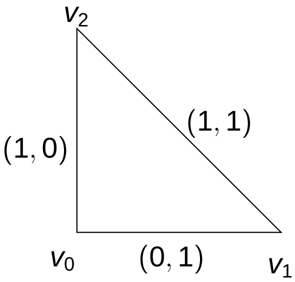
A characteristic function of $\triangle^n$:

$$\xi : \mathcal{F}(\triangle^n) \to \mathbb{Z}^n$$

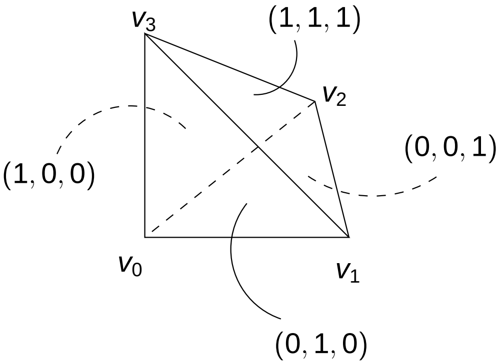
by

$$\xi(F_i) = e_i$$

Characteristic function of $\triangle^2$



Characteristic function of $\triangle^3$



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$$T_F = \mathbb{Z}_F(\mathbb{R})/\mathbb{Z}(F) \subset T^n$$

Davis and Januszkiewicz (1991) construction of $\mathbb{C}P^n$

Define $\sim$ on $T^n \times \triangle^n$ by

$$(t, x) \sim (s, y) \text{ iff } x = y \text{ and } st^{-1} \in T_F \text{ where } x \in F^0.$$

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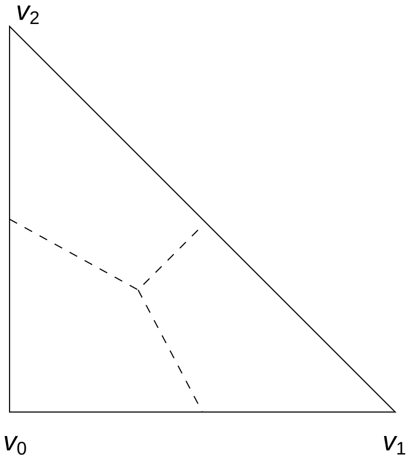
The orbit map

$$\pi : M(\triangle^n, \xi) \to \triangle^n$$

is given by

$$\pi([t, x]^{\sim}) \to x.$$

Cubical subdivision of $\triangle^2$



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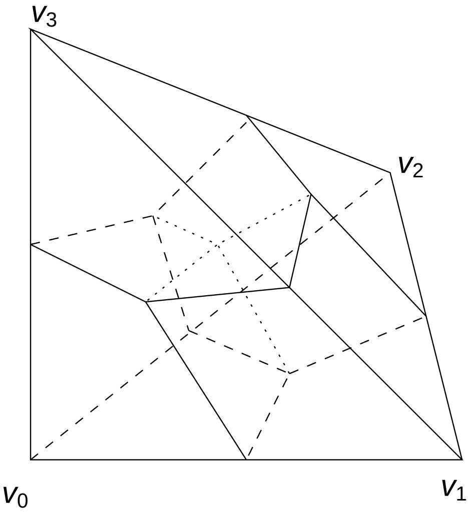

A subdivision of $\mathbb{C}P^n$

We have

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If σ is a face of τ then

$$\pi^{-1}(I_\tau) \subset \pi^{-1}(I_\sigma).$$