Complexity one GKM graph with symmetries and an obstruction to be a torus graph

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TOPOLOGY OF TORUS ACTIONS AND APPLICATIONS TO GEOMETRY AND COMBINATORICS
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GKM theory [Guillemin-Zara, 2000∼]

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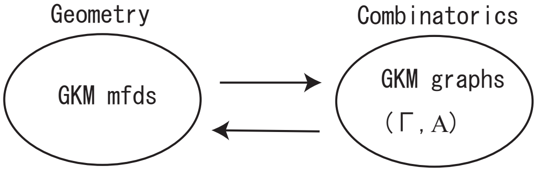

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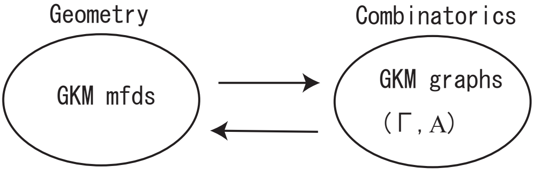

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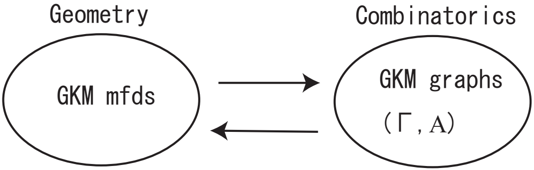

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Toric mfds, S^{2n} (torus mfds), G/H with $T^n \subset H \subset G$, ...

Problem

When does (M^{2m}, T^n) extend to (M^{2m}, T^ℓ) or (M^{2m}, G)?
Here, $\ell \geq n$ and G is a cpt Lie gr with $T^n \subset G$ (maximal).

(Abstract) GKM graph

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be an *m*-valent graph, i.e., $\#E_p(\Gamma) = m$ for all $p \in V(\Gamma)$.

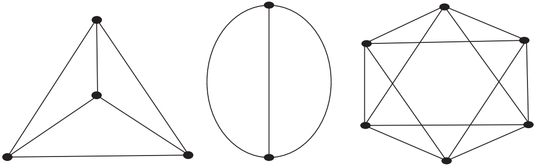

Figure: Two 3-valent graphs and one 4-valent graph.

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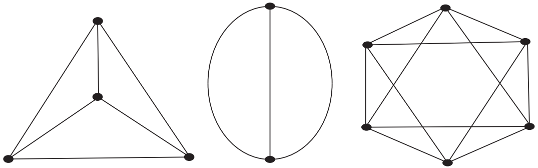


Figure: Two 3-valent graphs and one 4-valent graph.

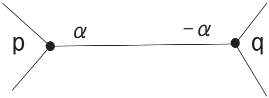
Definition

A (complex) GKM graph is a labelled graph $(\Gamma, \mathcal{A})$, where a label $\mathcal{A} : E(\Gamma) \to H^2(BT^n) \simeq \mathbb{Z}^n$ (for $1 \leq n \leq m$) satisfies the following conditions:

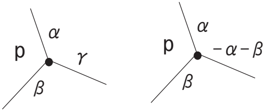
Axial function $\mathcal{A}$ (I)

$\mathcal{A} : E(\Gamma) \to H^2(BT^n) \simeq \mathbb{Z}^n$ (called axial function) satisfies the following 3 conditions:

(1) $\mathcal{A}(pq) = -\mathcal{A}(qp)$



(2) $\{\mathcal{A}(e) \mid e \in E_p(\Gamma)\}$ is pairwise linearly independent



where $H^2(BT^3) = \langle \alpha, \beta, \gamma \rangle$.

Axial function $\mathcal{A}$ (II) and Examples

(3) $\forall pq \in E(\Gamma)$, $\exists$ a bijection $\nabla_{pq} : E_p(\Gamma) \to E_q(\Gamma)$ which satisfies $\forall e \in E_p(\Gamma)$, $\exists c_{pq}(e) \in \mathbb{Z}$ s.t. $\mathcal{A}(\nabla_{pq}(e)) - \mathcal{A}(e) = c_{pq}(e)\mathcal{A}(pq)$.
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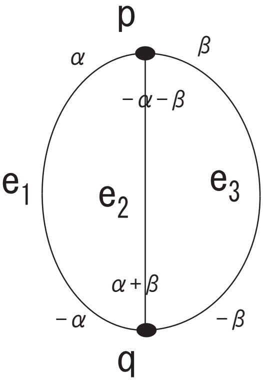

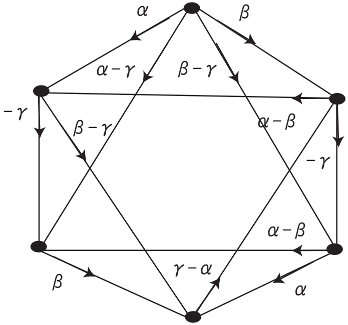

Problem from comb. point of view

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Problem (Combinatorial interpretation of $(M^{2m}, T^n) \Rightarrow (M^{2m}, T^\ell)$)

When does $\mathcal{A} : E(\Gamma) \to H^2(BT^n)$ ((m, n)-type) extend to $\widetilde{\mathcal{A}} : E(\Gamma) \to H^2(BT^\ell)$ ((m, ℓ)-type)?
Here, $n \leq \ell \leq m$.

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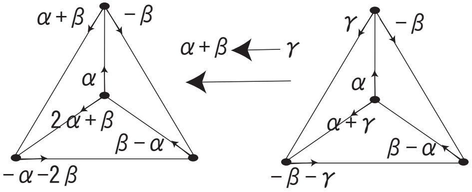

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The integer $c_{pq}(e)$ of the condition (3) does NOT change! Namely, $\forall e \in E_p(\Gamma)$, $\exists c_{pq}(e) \in \mathbb{Z}$ s.t.

$$\mathcal{A}(\nabla_{pq}(e)) - \mathcal{A}(e) = c_{pq}(e)\mathcal{A}(pq) \text{ for } (\Gamma, \mathcal{A}),$$

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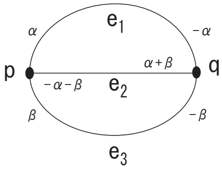
Thus, the map

$$c_{(\Gamma,\mathcal{A})} : E(\Gamma) \to \mathbb{Z}^m \quad \text{s.t.} \quad c_{(\Gamma,\mathcal{A})}(pq) = (c_{pq}(e_1), \ldots, c_{pq}(e_m))$$

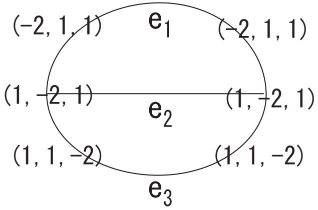
is <u>invariant under the extension!</u> (where $E_p(\Gamma) = \{e_1, \ldots, e_m\}$)

Example ($c_{(\Gamma,\mathcal{A})} : E(\Gamma) \to \mathbb{Z}^n$)

Let $(\Gamma, \mathcal{A})$ be the following $(3, 2)$-type GKM graph.



Then, the map $c_{(\Gamma,\mathcal{A})} : E(\Gamma) \to \mathbb{Z}^3$ is as follows:



Obstruction (Idea and definition)

IDEA and definition

By modified the definition of
the equivariant cohomology of GKM graph (Braden-MacPherson),
we define the following $\mathbb{Z}$-module from $c_{(\Gamma,\mathcal{A})} : E(\Gamma) \to \mathbb{Z}^m$:

$$\mathcal{O}(c_{(\Gamma,\mathcal{A})}) = \{f : V(\Gamma) \to \mathbb{Z}^m \mid \nabla_{pq}(f_p) - f_q = f_q(qp)c_{(\Gamma,\mathcal{A})}(qp)\}$$

where $f(p) = f_p \in \mathbb{Z}^m = \mathbb{Z}E_p(\Gamma)$ and $f_q(qp) \in \mathbb{Z}$ is an integer corresponding to the edge qp.

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Theorem (Obstruction)

1. $\mathcal{O}(c_{(\Gamma,\mathcal{A})})$ *is a free* $\mathbb{Z}$*-module with* $n \leq \mathrm{rk}\mathcal{O}(c_{(\Gamma,\mathcal{A})}) \leq m$;
2. $\exists$ *an* (m, ℓ)*-type extension* $\overset{iff}{\Longleftrightarrow} \ell \leq \mathrm{rk}\mathcal{O}(c_{(\Gamma,\mathcal{A})})$.

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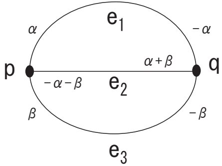

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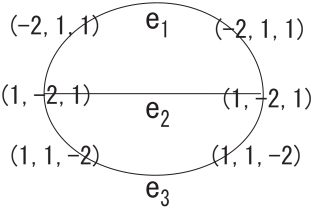

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$$= \{(f_p, f_q) = ((x,y,z),(-x,-y,-z)) \mid x+y+z=0\} \simeq \mathbb{Z}^2.$$

Therefore, $\mathrm{rk}\mathcal{O}(c_{(\Gamma,\mathcal{A})}) = 2(< 3)$.

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Remark

We can also prove this result by using Shunji Takuma's obstruction (2004).

Application

Let $(\Gamma, \mathcal{A})$ be an (m, n)- type GKM graph $(m \geq n)$.

Definition

If $m = n$, we call $(\Gamma, \mathcal{A})$ a torus graph.
If $m = n + 1$, we call $(\Gamma, \mathcal{A})$ a complexity one GKM graph.

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Assumption

Assume that the complexity one GKM graphs have (rank 2, 3) Weyl group actions $(S_3, S_4, S_2^{\pm}, S_3^{\pm}, D_6)$.

Motivation of the assumption.

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Proposition

If (M^{2n+2}, T^n) extends to (M, G) s.t. $M = G/H$, then $n = 2$ or 3. If $n = 2$, then

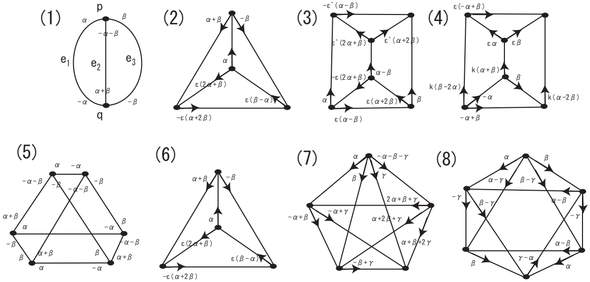
(1) $\mathcal{F}\ell(\mathbb{C}^3) \cong SU(3)/T^2$, (2) $Q_3 \cong SO(5)/SO(3) \times SO(2)$,
(3) $\mathbb{C}P^3 \cong Sp(2)/Sp(1) \times T^1$, (4) $S^6 \cong G_2/SU(3)$.

If $n = 3$, then

$$G_2(\mathbb{C}^4) \cong SU(4)/S(U(2) \times U(2)).$$

Main Theorem

If $(\Gamma, \mathcal{A})$ be a $3,4$-valent complexity one GKM graph with $S_3, S_4, S_2^{\pm}, S_3^{\pm}$ or D_6 actions. Then, $(\Gamma, \mathcal{A})$ is one of the followings (classification):



Furthermore, $(\Gamma, \mathcal{A})$ extends to the torus graph $\stackrel{iff}{\Longleftrightarrow}$ (2), (3), (4), (7) (extension).

Remark

There are the following geometric models for each GKM graph in Main Thm:

- $(1) = (S^6, G_2)$,
- $(2) = (\mathbb{C}P^3, SU(3))$,
- $(3) = (\mathbb{C}P^3 \# \mathbb{C}P^3, SU(3))$,
- $(4) = (S^5 \times_{T^1} \mathbb{P}(\gamma_k \oplus \underline{\mathbb{C}}), SU(3))$,
- $(5) = (\mathcal{F}l(\mathbb{C}^3), SU(3))$,
- $(6) = (Q_3, SO(5))$,
- $(7) = (\mathbb{C}P^4, SU(4))$,
- $(8) = (G_2(\mathbb{C}^4), SU(4))$.

Thank you for your attention

Happy 60th Birthday,
Prof. Mikiya Masuda and Prof. Dong Youp Suh!